An Application of the Infinite Canonical Ramsey Theory to Geometry

1 Introduction

Let $X \subseteq \mathsf{R}^2$ be infinite. Is there an infinite $Y \subseteq X$ such that all pairs of points in Y have distinct distances? We will show YES using the Infinite Can Ramsey Theorem! We will then move onto R^d where some issues will arise.

Our proofs have two ingredients: (1) the canonical Ramsey numbers, and (2) geometric lemmas about points in R^d.

Definition: Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathsf{N}$.

- V is *homogenous* if $COL(a,b) = COL(c,d)$ iff $TRUE$.
- V is *min-homogenous* if $COL(a,b) = COL(c,d)$ iff $a = c$.
- V is *max-homogenous* if $COL(a,b) = COL(c,d)$ iff $b = d$.
- V is *rainbow* if $COL(a,b) = COL(c,d)$ iff $a = c$ and $b = d$.

Recall the Infinite Canonical Ramsey Theorem:

Theorem 1.1 *Let $COL : \binom{\mathsf{N}}{2} V \rightarrow \omega$. Then one of the following occurs.*

- *There exists an infinite homog set.*
- *There exists an infinite min-homog set.*
- *There exists an infinite max-homog set.*
- *There exists an infinite rainbow set.*

2 Points in R^2 and Distance

Theorem 2.1 *If $X \subseteq \mathsf{R}^2$ is an infinite set then there exists infinite $Y \subseteq \mathsf{R}^2$ such that all the distances between points of Y are distinct.*

Proof: Let $X = \{p_1, p_2, p_3, \ldots\}$. Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ be defined as

$$COL(i,j) = |p_i - p_j|.$$

Apply the infinite canonical Ramsey Theorem to this coloring.

If there is an infinite rainbow set we are done. We show that none of the other cases can occur.

If there is an infinite homogenous set then there exists an infinite set of points such that all the distances between them are distinct. Let this set be

$$\{q_1, q_2, q_3, q_4, q_5 \ldots\}.$$

(There are many ways to show this is a contradiction. We do one of them.) The points q_1, q_2, q_3 form an equilateral triangle. q_4 has no place to go.

If there is an infinite min-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the lower indexed point. Let this set be

$$\{q_1, q_2, q_3, q_4, q_5 \ldots\}.$$

The points $q_3, q_4, \ldots$ are all on a circle centered at q_1.

The points $q_3, q_4, \ldots$ are all on a circle centered at q_2.

Both of these circles have $q_4, q_5, q_6, \ldots$ on them. If two circles intersect in ≥ 3 points then they are the same circle, hence these two circles are the same. Hence their centers agree so $q_1 = q_2$.

If there is an infinite max-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the higher indexed point. Let this set be

$$\{q_1, q_2, q_3, q_4, q_5 \ldots\}.$$

The points $q_1, q_2, q_3, \ldots$ are all on a circle centered at q_4.

The points $q_1, q_2, q_3, \ldots$ are all on a circle centered at q_5.

If two circles intersect in ≥ 3 points then they are the same circle, hence these two circles are the same. Hence their centers agree so $q_4 = q_5$.

If there is an infinite max-homogenous set then there exists an infinite set of points ∎

3 Points in R^3 and Distance

We will try to prove the following theorem but hit a roadblock.

Theorem 3.1 *If $X \subseteq \mathsf{R}^3$ is an infinite set then there exists infinite $Y \subseteq \mathsf{R}^3$ such that all the distances between points of Y are distinct.*

Proof: Let $X = \{p_1, p_2, p_3, \ldots\}$. Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ be defined as

$$COL(i, j) = |p_i - p_j|.$$

Apply the infinite canonical Ramsey Theorem to this coloring.

If there is an infinite rainbow set we are done. We show that none of the other cases can occur.

If there is an infinite homogenous set then there exists an infinite set of points such that all the distances between them are distinct. Let this set be

$$\{q_1, q_2, q_3, q_4, q_5 \ldots\}.$$

(There are many ways to show this is a contradiction. We do one of them.) The points q_1, q_2, q_3, q_4 form an equilateral tetrahedron. q_5 has no place to go.

If there is an infinite min-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the lower indexed point. Let this set be

$$\{q_1, q_2, q_3, q_4, q_5 \ldots\}.$$

The points $q_3, q_4, \ldots$ are all on a sphere centered at q_1. Now what? We need the theorem for the sphere and for the plane. Lets start over again

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We need an induction on R^d and S^d.

Def 3.2 The sphere S^d of radius r is: $S^d = \{x \in \mathsf{R}^{n+1} : |x| = r\}$.

Theorem 3.3 *For all $d \geq 1$ the following hold:*

1. *If $X \subseteq \mathsf{R}^d$ is infinite then there exists infinite $Y \subseteq X$ such that all distances between points of Y are distinct.*

2. *If $X \subseteq S^d$ is infinite then there exists infinite $Y \subseteq X$ such that all distances between points of Y are distinct. (We measure the distance in R^{d+1}. For example when looking at S^1 we look at chords of the circle.)*

Proof: We prove this by induction on d.

For $d = 1$:

Assume true for $d - 1$.

Let $X \subseteq \mathsf{R}^d$ be infinite. Let $X = \{p_1, p_2, \ldots\}$. We will use this ordering when applying Can Ramsey. Define the coloring $COL(i, j) = |p_i - p_j|$. Apply the Can Ramsey Theorem. We show that there cannot be an infinite homog, min-homog, or max-homog set.

Assume by way of contradiction that X be an infinite homog set. So all of the points in X are equidistant from each other. One CAN show that X must be finite with geometry, however I'll just use the ind hyp. Let $p \in X$. Then all of the other points are equidistant from p. They are on an S^{d-1} sphere. Apply the induction hypothesis to S^{d-1} to get the desired infinite set.

Let X be a min-homog set. By renumbering Let $X = \{p_1, p_2, \ldots\}$. We know that $|p_1 - p_2| = |p_1 - p_3| = \cdots$. Then all of the other points are equidistant from p. They form an S^{d-1} sphere. Apply the induction hypothesis to S^{d-1} to get the desired infinite set.

Let X be a max-homog set. By renumbering Let $X = \{p_1, p_2, \ldots\}$. We know that $|p_1 - p_3| = |p_2 - p_3|$ and $|p_1 - p_4| = |p_2 - p_4|$ etc. Hence the points $\{p_3, p_4, \ldots\}$ are all equidistance between p_1 and p_2. So $\{p_3, p_4, \ldots\}$ are all on (with renaming) R^{d-1}. Then all of the other points are equidistant from p. They are on a space equivalent to an R^{d-1} sphere. Apply the induction hypothesis to S^{d-1} to get the desired infinite set.

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4 Lets have distinct Areas

We want to prove a theorem along these lines:

If $X \subseteq \mathsf{R}^2$ is infinite then there exists infinite $Y \subseteq X$ such that all areas between 3-sets of points are distinct.

but this is false: just take the set of points on a line. Hence we look at points with no three colinear, often called *in general position*

First we need the 3-ary Can Ramsey theorem.

Def 4.1 Let $COL : \binom{\mathsf{N}}{3} \rightarrow \omega$. Let $I \subseteq \{1,2,3\}$. A set is *I-homog* if, for all $x_1 < x_2 < x_3$, $y_1 < y_2 < y_3$.

$$COL(x_1, x_2, x_3) = COL(y_1, y_2, y_3) \text{ iff } (\forall i \in I)[x_i = y_i].$$

Theorem 4.2 *For all $COL : \binom{\mathsf{N}}{3} \rightarrow \omega$ there exists $I \subseteq [3]$ and infinite $H \subseteq \mathsf{N}$ such that H is I-homog.*

We will assume it.

Theorem 4.3 *If $X \subseteq \mathsf{R}^2$ is infinite and in general position then there exists infinite $Y \subseteq X$ such that all the areas of 3-sets are different.*

Proof:

Let $X = \{p_1, p_2, \ldots\}$ and we use this ordering for Can Ramsey. Let

$COL(p_i, p_j, p_k)$ be the area of the triangle p_i, p_j, p_k.

Apply the 3-ary Can Ramsey. We show that of the 8 different kinds of homog, only rainbow (that is when $I = \{1,2,3\}$) is possible.

By renumbering let $H = \{p_1, p_2, \ldots\}$ for whichever type of homog it is

Let Y be $\emptyset$-homog. Then any of the cases below will show that there are three points colinear.

Let Y be $\emptyset$-homog OR 1-homog OR 2-homog OR 12-homog Then $AREA(p_1, p_2, p_k)$ is always the same. Hence p_k lies on one of TWO lines parallel to p_1p_2. So of p_3, p_4, p_5, p_6, p_7 three of them will be on the same line.

Let Y be 3-homog OR 23-homog Then $AREA(p_i, p_6, p_7)$ with $1 \le i \le 5$ is always the same. Hence $1 \le i \le 5$ all lie on one of two lines parallel to p_6, p_7. Hence there must be three on a line.

Let Y be 13-homog. Then $AREA(p_1, p_j, p_7)$ with $2 \le j \le 6$ is always the same. Hence $2 \le i \le 6$ all lie on one of two lines parallel to p_1, p_7. Hence there must be three on a line.
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