An Application of the Infinite Canonical Ramsey Theory to Geometry

1 Introduction

Let $X \subseteq \mathbb{R}^2$ be infinite. Is there an infinite $Y \subseteq X$ such that all pairs of points in $Y$ have distinct distances? We will show YES using the Infinite Can Ramsey Theorem! We will then move onto $\mathbb{R}^d$ where some issues will arise.

Our proofs have two ingredients: (1) the canonical Ramsey numbers, and (2) geometric lemmas about points in $\mathbb{R}^d$.

Definition: Let $COL : \binom{\mathbb{N}}{2} \to \omega$. Let $V \subseteq \mathbb{N}$.

- $V$ is homogenous if $COL(a, b) = COL(c, d)$ iff TRUE.
- $V$ is min-homogenous if $COL(a, b) = COL(c, d)$ iff $a = c$.
- $V$ is max-homogenous if $COL(a, b) = COL(c, d)$ iff $b = d$.
- $V$ is rainbow if $COL(a, b) = COL(c, d)$ iff $a = c$ and $b = d$.

Recall the Infinite Canonical Ramsey Theorem:

Theorem 1.1 Let $COL : \binom{\mathbb{N}}{2} \to \omega$. Then one of the following occurs.

- There exists an infinite homog set.
- There exists an infinite min-homog set.
- There exists an infinite max-homog set.
- There exists an infinite rainbow set.

2 Points in $\mathbb{R}^2$ and Distance

Theorem 2.1 If $X \subseteq \mathbb{R}^2$ is an infinite set then there exists infinite $Y \subseteq \mathbb{R}^2$ such that all the distances between points of $Y$ are distinct.

Proof: Let $X = \{p_1, p_2, p_3, \ldots\}$. Let $COL : \binom{\mathbb{N}}{2} \to \omega$ be defined as

$$COL(i, j) = |p_i - p_j|.$$  

Apply the infinite canonical Ramsey Theorem to this coloring.

If there is an infinite rainbow set we are done. We show that none of the other cases can occur.

If there is an infinite homogenous set then there exists an infinite set of points such that all the distances between them are distinct. Let this set be
\{q_1, q_2, q_3, q_4, q_5 \ldots \}.

(There are many ways to show this is a contradiction. We do one of them.) The points $q_1$, $q_2$, $q_3$ form an equilateral triangle. $q_4$ has no place to go.

If there is an infinite min-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the lower indexed point. Let this set be

\{q_1, q_2, q_3, q_4, q_5 \ldots \}.

The points $q_3, q_4, \ldots$ are all on a circle centered at $q_1$.
The points $q_3, q_4, \ldots$ are all on a circle centered at $q_2$.

Both of these circles have $q_4, q_5, q_6, \ldots$ on them. If two circles intersect in $\geq 3$ points then they are the same circle, hence these two circles are the same. Hence their centers agree so $q_1 = q_2$.

If there is an infinite max-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the higher indexed point. Let this set be

\{q_1, q_2, q_3, q_4, q_5 \ldots \}.

The points $q_1, q_2, q_3, \ldots$ are all on a circle centered at $q_4$.
The points $q_1, q_2, q_3, \ldots$ are all on a circle centered at $q_5$.

If two circles intersect in $\geq 3$ points then they are the same circle, hence these two circles are the same. Hence their centers agree so $q_4 = q_5$.

If there is an infinite max-homogenous set then there exists an infinite set of points

\[\textbf{3 Points in } \mathbb{R}^3 \text{ and Distance}\]

We will try to prove the following theorem but hit a roadblock.

\textbf{Theorem 3.1} If $X \subseteq \mathbb{R}^3$ is an infinite set then there exists infinite $Y \subseteq \mathbb{R}^3$ such that all the distances between points of $Y$ are distinct.

\textbf{Proof:} Let $X = \{p_1, p_2, p_3, \ldots \}$. Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ be defined as

\[COL(i, j) = |p_i - p_j|\.

Apply the infinite canonical Ramsey Theorem to this coloring.

If there is an infinite rainbow set we are done. We show that none of the other cases can occur.

If there is an infinite homogenous set then there exists an infinite set of points such that all the distances between them are distinct. Let this set be
\{q_1, q_2, q_3, q_4, q_5 \ldots \}.

(There are many ways to show this is a contradiction. We do one of them.) The points $q_1, q_2, q_3, q_4$ form an equilateral tetrahedron. $q_5$ has no place to go.

If there is an infinite min-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the lower indexed point. Let this set be

\{q_1, q_2, q_3, q_4, q_5 \ldots \}.

The points $q_3, q_4, \ldots$ are all on a sphere centered at $q_1$.
The points $q_3, q_4, \ldots$ are all on a sphere centered at $q_2$.
If two spheres intersect in $\leq 3$ points then SO WHAT! The intersection of two spheres is a circle. CAN have an infinite number of points on it.

If there is an infinite max-homogenous set then there exists an infinite set of points such that all the distances between them all depend on the the higher indexed point. Let this set be

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If two circles intersect in $\leq 3$ points then SO WHAT! The intersection of two spheres is a circle. CAN have an infinite number of points on it.

SO, HOW TO FINISH? here are a few thoughts.

1. In the min-homog case we could have 80,000 spheres whose intersection is infinite. Surely this gives a contradiction. But the geometry is hard to visualize and may get even harder when we go to $\mathbb{R}^d$.

2. In the max-homog case we could have 80,000 spheres whose intersection is large (but finite). Surely this gives a contradiction. But the geometry is hard to visualize and may get even harder when we go to $\mathbb{R}^d$.

3. We could first prove the theorem on the circle! Then if we get a circle we use the circle-theorem. Not sure what to do about the max-homog case.

4. Prove it without using Can Ramsey. Maybe we could do this but it would make Bill sad – unless it was his proof.
ERIK- WHEN I first did this proof with Sam Zbarsky many years ago we did the alternation between proving it in $\mathbb{R}^d$ and $S^d$ (d-sphere- might be off-by-1 there) and we thought we had to. We were doing the finite version.

I now do not see if I can have a nice proof in the infinite case since max-homog only gives finite intersection.

So here are my questions:

• (I am sure the answer is YES but messy especially in $\mathbb{R}^d$): Fill in the X and Y so that the following is true: if the intersection of X d-spheres has Y points then all of the d-spheres are the same.

• Is there a way to finish this proof without using a theorem like that in my first question to you. (Actually the answer is YES in a funny way- the next item IS such a proof, my proof, which does not use Can Ramsey, so I am not sad.)

• Here is non-can-ramsey proof for $\mathbb{R}^d$ and please verify it, though this is only a sketch. First proof for $\mathbb{R}^1$ and $\mathbb{S}^1$ (the circle), the proof is similar to the on below but actually easier. Assume true for $\mathbb{R}^{d-1}$ and $\mathbb{S}^{d-1}$. (I may be off-by-1 on the $\mathbb{S}^d$.) So its a proof by induction.

We call a set with all distances distinct dist-rainbow. Let $X \subseteq \mathbb{R}^d$ be infinite set. Let $Y \subseteq X$ be a maximal dist-rainbow set. That means that Y is dist-rainbow but for all $z \notin Y$, $Y \cup \{z\}$ is NOT dist-rainbow. Assume, By Way of Contradiction, that $Y$ is finite. Let $Z = X - Y$. We define a mapping from $Z$ to $\binom{Y}{2} \cup Y \times \binom{Y}{2}$.

If $z \in Z - Y$ then WHY is $z \notin Y$? Either

- There exists two points $p, q \in Y$ such that $|z - p| = |z - q|$. Then map $z$ to $\{p, q\}$.
- There exists three points $p, q, r \in Y$ such that $|z - p| = |q - r|$. Then map $z$ to $(p, \{q, r\})$.

Since $Y$ is finite some element of $\binom{Y}{2} \cup Y \times \binom{Y}{2}$ is mapped to infinitely often.

If there exists $\{p, q\} \in \binom{Y}{2}$ that gets mapped to infinitely often then there is an infinite number of points of X that are equidistant from $p$ and $q$. AH- thats a line in $\mathbb{R}^{d-1}$ - so use induction.

If there exists $(p, \{q, r\}) \in Y \times \binom{Y}{2}$ that gets mapped to infinitely often then there is an infinite number of points of X that are equidistant from $p$ (and that distance is $|q - r|$ though we don’t need that). AH- thats $\mathbb{S}^{d-1}$ - so use induction.

To think about for later- does this same approach work for triangle-areas in $\mathbb{R}^2$ (NOTE- would have to make the points no-3-colinear). Or for points in gen position in $\mathbb{R}^d$ and you want distinct a-volumes?