Homework 9, Morally Due Tue Apr 24, 2018

1. (40 points) Use the Prob method to get a lower bound on $W(k,c)$.

SOLUTION TO PROBLEM TWO

We will determine N later.

c-color $\{1,\ldots,N\}$ randomly.

We bound the prob that there IS a mono k-AP.

The number of colorings is c^N.

The number of colorings that have a mono k-AP is bounded above as follows:

- Pick the first element of the mono sequence. This is done in any of $N-k$ ways, but we bound by N.
- Pick the difference d. This is done in any of N/k ways.
- Pick the color. This is done in any of c ways.
- Color everything else, This is done in c^{N-k} ways.

So the number of ways is bounded above by

$$N \times \frac{N}{k} \times c \times c^{N-k} = \frac{N^2 c^{N-k+1}}{k}$$

The prob of getting a mono k-AP is bounded above by

$$\frac{N^2 c^{N-k+1}}{kc^N} = \frac{N^2}{kc^{k-1}}$$

We seek a large N such that the above is < 1. Hence we need

$$N^2 < kc^{k-1}$$

$$N < \sqrt{k}c^{(k-1)/2}$$

END OF SOLUTION TO PROBLEM TWO

2. (40 points) In this problem an iso-L is a set of points that form an isosceles L. Formally an iso-L is (x, y), $(x + d, y)$, and $(x, y + d)$ where $x, y, d \in \mathsf{N}$.

   (a) (20 points) Use VDW theorem to show that there exists N such that, for all 2-colorings of $[N] \times [N]$ there exists a monochromatic isosceles L shape. How big is your N? (you may use VDW numbers in the expression).

   (b) (20 points) Use VDW theorem to show that there exists N such that, for all 3-colorings of $[N] \times [N]$ there exists a monochromatic isosceles L shape. How big is your N (you may use VDW numbers in the expression).

   (c) (0 points but think about) Use VDW theorem to show the following. For all c there exists $N(c)$ such that, for all c-colorings of $[N(c)] \times [N(c)]$ there exists a monochromatic isosceles L shape.

SOLUTION TO PROBLEM THREE

a) Let $N = W(3, 2)$. Let COL be a 2-coloring of $[N] \times [N]$. Look at the diagonal:

$$(1, N), (2, N - 1), \ldots, (N, 1)$$

Apply VDW's theorem to get a mono 3-AP which we assume is RED.

$$(a, N - a), (a + d, N - a - d), (a + 2d, N - a - 2d)$$

Look at: $(a, N - a), (a + d, N - a - d)$. If $(a, N - a - d)$ as RED then we are done.

Look at: $(a, N - a), (a + 2d, N - a - 2d)$. If $(a, N - a - 2d)$ as RED then we are done.

Look at: $(a + d, N - a - d), (a + 2d, N - a - 2d)$ If $(a + d, N - a - 2d)$ as RED then we are done.

The only case left is that the following are all BLUE:

$$(a, N - a - d), (a, N - a - 2d), (a + d, N - a - 2d)$$

And again we are done.

b) Assume the colors are RED, BLUE, GREEN. The idea is that if we use VDW's theorem then there will many equally spaced points that are (say) RED which will force a nice regular pattern of points cooled just BLUE and GREEN. We will then use the prior result.

Let $N = W(k+1, 3)$ where we determine k later (we use $k+1$ just for notational convenience). Let COL be a 3-coloring of $[N] \times [N]$. Look at the diagonal:

$$(1, N), (2, N-1), \ldots, (N, 1)$$

Apply VDW's theorem to get a mono k-AP which we assume is RED.

$$(a, N-a), (a+d, N-a-d), (a+2d, N-a-2d), \ldots, (a+kd, N-a-kd)$$

For every pair $1 \le i < j \le N-1$:

Look at $(a+id, N-a-id), (a+jd, N-a-jd)$. If $(a+id, N-a-jd)$ is RED then we are done since

$(a+id, N-a-jd)$:

Add $(j-i)d$ to the first coordinate to get $(a+jd, N-a-jd)$.

Add $(j-i)d$ to the second coordinate to get $(a+id, N-a-id)$.

SO we now have that, for all $0 \le i < j \le N$, $(a+id, N-a-id), (a+jd, N-a-jd)$ is either BLUE or GREEN, so its 2-colored.

This is the bottom part of the $[N-1] \times [N-1]$ grid. If $N-1$ is $W(3, 2)$ then we can use the last part of this problem to get an iso-L shape since that proof only used the diag bottom of the grid.

END OF SOLUTION TO PROBLEM THREE

3. (20 points) Prove that there is no solution to $x^4 + y^4 = z^4$ with $x, y, z \in \mathsf{N}$ (This is the $n = 4$ case of Fermat's last theorem. You may go to the web or a textbook or your local number theorist for help, but you must write it in your own words and understand what you write.) Did your proof use that the number of primes is infinite?