HW 07, Problem 4, Solution

May 10, 2020

The Language of $\leq 3 - ary$ Colored Hypergraphs

Our language has the following predicates

1. $R(x)$, $B(x)$. Implicity that every vertex is R or B or NEITHER.
2. $RR(x, y)$, $BB(x, y)$, $GG(x, y)$. Implicity that every edges is RR or BB or GG or NEITHER.
3. $RRR(x, y, z)$, $BBB(x, y, z)$. Implicity that every 3-edges is RRR or BBB or NEITHER.

We call this object a JAMIE.

Conventions

1. Symmetric. So $RR(x,y)$ really means $RR(x,y) \wedge RR(y,x)$. Similar for BB, GG, RRR, BBB.
2. No self loops, so $R(x,x)$ is always false. Simlar for...
3. $(\exists x_1) \cdots (\exists x_n)$ means they are DISTINCT.
4. $(\forall x_1) \cdots (\forall x_n)$ means they are DISTINCT.

Main Theorem

Theorem The following function is computable: Given ϕ, an E^*A^* sentence in the theory of JAMIE, output $\mathrm{spec}(\phi)$. ($\mathrm{spec}(\phi)$ will be a finite or cofinite set; hence it will have an easy description.)

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We will take ϕ to be

$$(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

Claim 1

Let $G \models \phi$ with witnesses $v_1, \ldots, v_n$. Let H be an induced subgraph of G that contains $v_1, \ldots, v_n$. Then $H \models \phi$.
Proof similar to the one from class.

Claim 2, The Main Claim

If $(\exists N \ge QQQ)[N \in \mathrm{spec}(\phi)]$ then

$$\{n+m, \ldots, QQQ, \ldots\} \subseteq \mathrm{spec}(\phi).$$

We will derive what QQQ has to be later.

Proof of Claim 2

Since $N \in \mathrm{spec}(\phi)$ there exists G, a JAMIE on N vertices such that $G \models \phi$. Let $v_1, \ldots, v_n$ be such that:

$$(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].$$

(Proof continued on next slide)

Proof of Claim 2 Continued

$$(\forall y_1)\cdots(\forall y_m)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)].$$

Let $X = \{v_1,\ldots,v_n\}$ and $U = V - X$. Note that $|U| \ge QQQ - n$.
We color U by how it relates to all of the elements in X:

1. For all $1 \le i \le n$ $RR(u,v_i)BB(u,v_i)GG(u,v_i)$ (≤ 8 options).
   There are n of them, so $8^n = 2^{3n}$ options.
2. For all $1 \le i < j \le n$ $RRR(u,v_i,v_j)BBB(u,v_i,v_j)$. ($\le 4$ options)
   There are $\binom{n}{2}$ of them, so $\le 4^{n^2/2} = 2^{n^2}$.

The number of colors is $2^{3n} \times 2^{n^2} = 2^{n^2+3n}$.

Proof of Claim 2 Continued

We want LOTS of elements to be the SAME color. So we want $\frac{QQQ-n}{2^{n^2+3n}}$ to be LARGE (and to be a natural number). So we let $QQQ = (L+n)2^{n^2+3n}$ where L will be determined later.

Every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^{n^2+3n}L$ there exists L vertices that map to the same color. Denote the L elements of U that map to the same color U'.

We denote the color they all map to as THECOLOR.

Proof of Claim 2 Continued

We thin out U' on this and the next two slides.

Some of the $u \in U$ have $R(u)$ true, some have $B(u)$ true, and some have neither.

At least $L/3$ of the U' have the same. We'll say its R.

Let U'' be al the $u \in U$ such that $R(u)$ holds.

We assume $U'' = L/3$, or $L = 3U''$.

Proof of Claim 2 Continued

(Erika says to apply Ramsey Theory here).

$\binom{U''}{2}$ is 4-colored by RR, BB, GG, NEITHER.

Let U''' be the homog set. Assume its NEITHER

We assume U'' big enough to yield a homog set of size U''' where we will figure out U''' later.

So $U'' = R_2(U''', 4)$, so $L = 3R_2(U''', 4)$.

Proof of Claim 2 Continued

$\binom{U'''}{3}$ is 3-colored by RRR, BBB. NEITHER.

Let U'''' be the homog set. Assume its GGG.

We assume U''' big enough to yield a homog set of size U'''' where we will figure out U'''' later.

So $U''' = R_3(U'''', 3)$, so $L = 3R_2(R_3(U'''', 3), 4)$.

We will need $U'''' = m$ so

$$L = 3R_2(R_3(m, 3), 4).$$

$$QQQ = (L + n)2^{n^2+3n} = (3R_2(R_3(m, 3), 4) + n)2^{n_3^2 n}$$

Let $U'''' = \{u_1, \ldots, u_m\}$.

Proof of Claim 2 Continued

Let H_0 be G restricted to $X \cup \{u_1, \ldots, u_m\}$. By Claim 1 $H_0 \models \phi$.
For every $p \geq 1$ we form a JAMIE H_p on $n + m + p$ vertices such that $H_p \models \phi$:
Informally add $m + p$ vertices that are **just like the u_i's**.
Formally Next Slide.

Proof of Claim 2 Continued, Formal $H_p = (V_p, E_p)$

$V_p = X \cup \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+p}\}$ where $u_{m+1}, \ldots, u_{m+p}$ are new vertices.
We have to define how the new u_i's relate to X, to the other u_is (both old and new).

- The new u_i's relate to the elements of X the same way the $\{u_1, \ldots, u_m\}$ did, which follows THECOLOR.
- For all $m+1 \le i \le m+p$, $R(u_i) = T$, $B(u_i) = F$.
- For all $1 \le i < j \le m+p$, NONE of $RR(u_i, u_j)$ are true.
- For all $1 \le i < j < k \le m+p$, $GGG(u_i, u_j, u_k) = T$.

X sees all of the $u_1, \ldots, u_{m+p}$ as the same. Hence any subset of the $\{u_1, \ldots, u_{m+p}\}$ of size m looks the same to X and to the other u_i's. Hence $H_p \models \phi$, so $n + m + p \in \mathrm{spec}(\phi)$.
End of Proof of Claim 2

THE REST OF THE PROOF

The rest of the proof is identical to what I did in class except that I replace $n + R(m)$ with QQQ.

Even so, its in the next slides.

Claim 3

$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$.
$N_0 = QQQ$.
$N_0 \notin \mathrm{spec}(\phi) \implies \mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}$.

Proof of Claim 3

By Claim 2
$\{N_0,\ldots\} \cap \mathrm{spec}(\phi) \neq \emptyset \implies \{n+m,\ldots,N_0,\ldots\} \subseteq \mathrm{spec}(\phi)$.
We take the contrapositive with a weaker premise.

$$N_0 \notin \mathrm{spec}(\phi) \implies \{N_0,\ldots\} \cap \mathrm{spec}(\phi) = \emptyset$$

$$\implies \mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}.$$

End of Proof of Claim 3

Recap Both Claims

We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
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$$\text{If } N_0 \in \mathrm{spec}(\phi) \text{ then } \{n+m, \ldots, \} \subseteq \mathrm{spec}(\phi).$$

Claim 3

$$\text{If } N_0 \notin \mathrm{spec}(\phi) \text{ then } \mathrm{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}.$$

Algorithm for Finding $\mathrm{spec}(\phi)$

1. Input

$$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)].$$

2. Let $N_0 = QQQ$. Determine if $N_0 \in \mathrm{spec}(\phi)$.

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2. Let $N_0 = QQQ$. Determine if $N_0 \in \text{spec}(\phi)$.

   2.1 If YES then by Claim 2 $\{n + m, \ldots\} \subseteq \text{spec}(\phi)$.
   For $0 \le i \le n + m - 1$ test if $i \in \text{spec}(\phi)$. You now know $\text{spec}(\phi)$ which is co-finite. Output it.

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2. Let $N_0 = QQQ$. Determine if $N_0 \in \mathrm{spec}(\phi)$.

   2.1 If YES then by Claim 2 $\{n+m,\ldots\} \subseteq \mathrm{spec}(\phi)$. For $0 \le i \le n+m-1$ test if $i \in \mathrm{spec}(\phi)$. You now know $\mathrm{spec}(\phi)$ which is co-finite. Output it.

   2.2 If NO then, by Claim 3 $\mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}$. For $0 \le i \le N_0-1$ test if $i \in \mathrm{spec}(\phi)$. You now know $\mathrm{spec}(\phi)$ which is finite set. Output it.

End of Proof of Main Theorem