What do you want to See Covered/Not Covered Next Time I teach This

Please email me a list of which topics you would have wanted to see covered. You can list as many as you want, and you can give commentary.

VDW Topics

1. **Poly VDW Thm** For all $p_1, \ldots, p_k \in \mathsf{Z}[x]$ such that $p_i(0) = 0$ for all $i \in [k]$, and $c \in \mathsf{N}$, there exists $W = W(p_1, \ldots, p_k; c)$ such that for all COL: [W] $\rightarrow$ [c] there exists a, d

$$a, a + p_1(d), \ldots, a + p_k(d) \text{ all the same color}$$

   Uses VDW.

2. **Can VDW** For all k there exists $W = W(k)$ such that for any COL: [W] $\rightarrow$ $[\omega]$ there exists a, d such that either

$$a, a + d, \ldots, a + (k-1)d \text{ are all the same color}$$

   or

$$a, a + d, \ldots, a + (k-1)d \text{ are all different colors}$$

   Can VDW uses 2-dim VDW. Extends to a-dim VDW uses $2a$-dim VDW That might be on a HW or midterm.

3. **LEGIT APPLICATION To Number Theory**

   - For all k there exists p_o such that for all primes $p \geq p_o$ there are k consecutive squares mod p.
   - For all k there exists p_o such that for all primes $p \geq p_o$ there are k consecutive non-squares mod p.

4. **Folkman's Thm** For all k, c there exists $N = N(k, c)$ such that for all COL: [N] $\rightarrow$ [c] there exists $x_1, \ldots, x_k$ such that ALL non-empty sums of the x_i's are the same color.

   Uses VDW.

5. **Hilbert's Cube Lemma** For all k, c there exists $H = H(k,c)$ such that for all COL: [H] $\rightarrow$ [c] there exists $x_0, x_1, \ldots, x_k$ such that

$$\{x_0 + \sum_{i=1}^{k} b_i x_i : b_i \in \{0,1\}\}$$

is monochromatic.

Can prove from VDW or directly.

6. **LEGIT APP TO REAL MATH!** EARLIEST Ramseyian Theorem Ever! Use HCL to prove

H Irreducibility Thm (2 var case). If $p(x,y) \in \mathsf{Q}[x,y]$ is irred then there exists $a \in \mathsf{Z}$ such that $p(x,a) \in \mathsf{Q}[x]$ is irred.

7. **Roth's Thm** Every set of upper positive density has a 3-AP.

There are three proofs of this: Combinatorial, Analytic, and Computer-assisted.

The idea of doing the Analytic Proof appeals to me since it would be a DIFFERENT proof technique then others that you seen. Non-Elementary proof, but not that hard.

8. **APPLICATION TO NUMBER THEORY** Schur's Theorem is a special case or Rado's Theorem.

Schur's Thm For all c there exists $S = S(c)$ such that for all COL: [S] $\rightarrow$ [c] there exists x, y, z same color such that $x + y = z$.

FLT For all $n \geq 3$ there does not exists $x, y, z \in \mathsf{N}$ such that $x^n + y^n = z^n$. (The $n = 4$ case was done by Fermat.)

Thm (Schur's Thm + FLT(4) implies there are an infinite number of primes.

9. **Rado's Theorem Over the Reals: True? False? A Matter of taste?** The following are equivalent

   - For all $COL\colon \mathsf{R} \rightarrow \mathsf{N}$ there exists mono w, x, y, z with $w+x = y+z$.
   - There is a cardinality between countable and the reals.

So the statement is Ind of ZFC.

10. **Hindman's Thm** For any finite coloring of N there exists an infinite A such that all finite sums of elements of A are the same color.

    Proof uses Ultrafilters so would be a DIFFERENT proof.

11. **Gallai-Witt Thm** Multi-dim VDW. I can prove this without using Hales-Jewitt.

12. **Hales-Jewitt Thm** A Very General Thm from which we can derive cleanly VDW and Gallai-Witt.

Ramsey Topics

1. **Ramsey Theory With Other Graphs** $R(C_k)$ is least n such that for all 2-coloring of $\binom{[n]}{2}$ there exists monochromatic k-cycle.

$$R(C_k) = \begin{cases} 6 & \text{if } k = 3 \text{ or } k = 4 \\ 2k - 1 & \text{if } k \geq 5 \text{ and } k \equiv 1 \pmod 2 \\ \frac{3k}{2} - 1 & \text{if } k \geq 4 \text{ and } k \equiv 0 \pmod 2 \end{cases} \quad (1)$$

   Would need to read the proofs to see how interesting they are.

2. **Ramsey Games** Example: Parameter k, n. Players RED and BLUE alternate coloring the edges of K_n. RED goes first. The first player to get a C_k in their color wins. For which n does RED have a winning strategy? Active Research.

3. **Thm** $R_3(k) \leq 2^{2^{4k}}$.

   Better is known:

   Thm $R_3(k) \leq 2^{2^{2k}}$.

   Would need to reread the proofs to see if I really could do it.

4. **Thm** $CR(k) \leq 2^{O(k^2 \log k)}$.

   Proof is Mileti-style.

5. **Large Can Ramsey** For all k there exists $n = n(k)$ such that for all COL: $\binom{\{k,\ldots,n\}}{2} \rightarrow [\omega]$ there is a large set that is either homog, min-homog, max-homog, rainbow.

   I have a general theorem that has inf-can, fin-can, large-can as corollaries. Its a mild reworking of the proof of Can Ramsey using 4-hypergraph Ramsey.

6. *a*-**ary Can Ramsey** Thm For all $a, k \in \mathsf{N}$ there exist $C = C(a, k)$ such that for all COL: $[\binom{[C]}{a}] \rightarrow [\omega]$ there exists a set H, $|H| = k$ and $1 \leq i_1 < \cdots < i_L \leq a$ such that for all $p_1 < \cdots < p_a \in H$ and $q_1 < \cdots < q_a \in H$

$$\mathrm{COL(p_1, \ldots, p_a) = COL(q_1, \ldots, q_a) \text{ iff } (p_{i_1}, \ldots, p_{i_L}) = (q_{i_1}, \ldots, q_{i_L})}$$

Similar to the proof on graphs, but messier. Some interesting upper bounds for $a = 3$ case, might just do that.

7. **Euclidean Ramsey Theory Sample Theorems**

   - If you c-color the plane there exists 2 points an inch apart same color. True for $c = 2, 3, 4$ (result for 4 is recent and computer-assisted). False for $c = 7$. Lots of literature on this.
   - Let T be a triangle with a 30, 90, or 150 degree angle. For every 2-coloring of R^2 there exists three points that form triangle T (note-actually form T, not just similar to T) that are monochromatic.

8. **Ramsey Multiplicity**

   Thm For all 2-col of K_n, exists $\frac{n^3}{24} - O(n^2)$ mono K_3's.

   This is the first thm in a field called **Ramsey Multiplicity**

   Here is the second thm

   Thm Fix k. For large n, for all 2-colorings of K_n there exists $\frac{n^2}{4^{k^2(1+o(1))}}$ mono K_k's.

Ramsey Theory and Logic, Ramsey Theory and Complexity

1. **Def** L is a language. Game:

   - Alice is Poly time and she has x, $|x| = n$.
   - Bob is all powerful and he has nothing.
   - They cooperate to determine if $x \in L$. Alice could just send Bob x. That takes n bits.

   Let L be the set of all 3-colorable graphs (or any NPC graph problem). Note size is $O(n^2)$. If there is a protocol in $O(n^{2-\epsilon})$ bits then $PH = \Sigma_2^p$. Proof used large 3-free set.

2. **Thm** For every computable COL: $\binom{\mathsf{N}}{2} \rightarrow [2]$ there is a Π_2-homogenous set. There is a computable coloring such that no homog set is Σ_2.

   Could do the direction: Given a computable coloring there is a Π_2-homog set.

   Really could not do the construction of a coloring, takes us too far afield.

3. **Def** $G \rightarrow (H_1, H_2)$ means that for every 2-coloring of the edges of G there is either a RED H_1 or a BLUE H_2.

   Marcus Schaefer proved the following.

   Thm $\{(G, H_1, H_2) : G \rightarrow (H_1, H_2)$ is Π_2^p-complete.

4. **Grid Color Extension (GCE)** is the set of tuples (n, m, c, χ) such that the following hold:

   - $n, m, c \in \mathsf{N}$. χ is a partial c-coloring of $[n] \times [m]$ that is rectangle-free.
   - χ can be extended to a rectangle-free coloring of $[n] \times [m]$.

   Thm GCE is NP-complete

5. **Def** Resolution proofs are a proof system to show that a Boolean Formula is NOT satisfiable. It is of interest to find a class of non-satisfiable formulas ϕ_n that require (say) $(1.5)^n$ long Res Proofs.

Def A graph is c-random if it does not contain a clique or ind set of size $c \log n$.

Def $\phi_{n,c}$ is a Boolean Formula that says **every** graph on n vertices is c-random. (This is false for c around $\frac{1}{2}$.)

Lauria, Pudlak, Rodl, Thapen proved:

Thm For appropriate c, any resolution proof for $\phi_{n,c}$ requires length $n^{\Omega(\log n)}$.

6. **Def** If for all $\mathrm{COL}\colon \binom{\kappa}{2}$ there is a homog set of size κ then κ is **Ramsey**.

   I could look into this and see what other theorems *of interest* follow from the existence of Ramsey Cardinals.

7. **Borel Colorings** Restrict the type of coloring of the reals and you can get some results.