Short Notes March 31st Lecture

1 Introduction

These notes are helpful if you both watched the recording and attended class (by zoom). Otherwise I doubt they are helpful.

Convention 1.1 Every time we mention a set of points in $\mathbb{R}^2$ they have no three colinear

2 Happy Ending Theorem

Def 2.1 Let $A \subseteq \mathbb{R}^2$ of size k. The points in A form a *convex k-gon* if for every $x, y, z \in A$, there is no point of A in the triangle formed by x, y, z. Henceforth we just say *k-gon*.

Theorem 2.2 *(Esther Klein) For every 5 points in $\mathbb{R}^2$ there exists a 4-gon.*

Theorem 2.3 *(Erdös and Szekeres) For all $k \geq 3$ there exists n such that for every set of n points in $\mathbb{R}^2$ there exists k of them that form a k-gon.*

Sketch:

$k = 3$: Take $n = 3$.

$k = 4$: Take $n = 5$ and use Klein's Theorem.

We assume $k \geq 5$.

We went over three proofs that used the following three colorings.

The points are $p_1, \ldots, p_n$. The ordering on the points is arbitrary; however, for the third proof we need the ordering.

Proof 1: $n = R_4(k)$. We have any n points in $\mathbb{R}^2$

$COL(w, x, y, z)$ is RED if the for points form a 4-gon, and BLUE if they do not.

The homog set can't be BLUE since if was then there would be $k \geq 5$ points such that NO 4-subset was a 4-gon, which contradicts Klein's Theorem.

Hence there are k points so that every set of 4 of them forms a 4-gon. One can show that the entire set is a k-gon.

Proof 1': We can use $n = R_4(k, 5)$ which is the smallest n such that any 2-coloring of $\binom{[n]}{4}$ has either a RED Homog set of size k or a BLUE homog set of size 5.

Proof 2: $n = R_3(k)$. We have any n points in $\mathbb{R}^2$

$COL(w, x, y)$ is RED if their is an EVEN number of points inside the x, y, z triangle, BLUE otherwise.

Both cases are possible. One can show that in either case the set is a k-gon using a parity argument.

Proof 3: $n = R_3(k)$. We have any n points in $\mathbb{R}^2$

$COL(p_i, p_j, p_k)$ where $i < j < k$ is RED if p_i, p_j, p_k is clockwise, and BLUE if counterclockwise.

Some cases, finishing the proof will be on the HW I give out next Tuesday.

∎

These bounds are quite large. The following upper and lower bounds are known.

Theorem 2.4

1. *(Erdös and Szekeres) For all* $k \geq 3$ *there exists* $n \leq \binom{2n-4}{n-2} + 1 = 4^{n+o(n)}$ *such that for every set of* n *points in* $\mathbb{R}^2$ *there exists* k *of them that form a* k*-gon.*

2. *(Andrew Suk) For all* $k \geq 3$ *there exists* $n \leq 2^{n+o(n)}$ *such that for every set of* n *points in* $\mathbb{R}^2$ *there exists* k *of them that form a* k*-gon.*

3. (a) *For all sets of 3 points in* $\mathbb{R}^2$ *there exists a subset of 3 that form a 3-gon (this is trivial). This is tight.*

   (b) *For all sets of 5 points in* $\mathbb{R}^2$ *there exists a subset of 4 that form a 4-gon. This is tight.*

   (c) *For all sets of 9 points in* $\mathbb{R}^2$ *there exists a subset of 5 that form a 5-gon. This is tight.*

   (d) *For all sets of 17 points in* $\mathbb{R}^2$ *there exists a subset of 6 that form a 6-gon. This is tight.*

4. *For all* $k \geq 3$ *there exists a set of* 2^{k-2} *points such that there is NO subset of size* k *that form a* k*-gon.*

The lower bound in the last part of the last theorem is the conjecture.

Conjecture 2.5 *For all $k \geq 3$ for every set of $2^{k-2}+1$ points in $\mathbb{R}^2$ there exists k of them that form a k-gon.*

3 Extends to Higher Dimensions

This was also on the Wikipedia Page of *The Happy Ending Problem*, so even though I just thought of it on the morning of March 31, it was somewhat studied. I am not surprised. But its gotten A LOT less attention than the planar case. In fact, I could not find it anywhere else on the web. If you can then let me know.

Convention 3.1 Every time we mention a set of points in $\mathbb{R}^3$ they have no four coplanar.

Def 3.2 Let $A \subseteq \mathbb{R}^3$ of size k. The points in A form a *convex k-gon* if for every $w, x, y, z \in A$, there is no point of A is in the tetrahedron formed by w, x, y, z. Henceforth we just say *k-gon*.

Theorem 3.3 *(Gasarch the Morning of March 31, but others many years ago) For all $k \geq 3$ there exists n such that for every set of n points in $\mathbb{R}^3$ there exists k of them that form a k-gon.*

Sketch:

$k = 3$: Take $n = 3$.

$k = 4$: Take $n = 5$ and use Klein's Theorem.

We assume $k \geq 5$.

We went over three proofs that used the following three colorings.

The points are $p_1, \ldots, p_n$. The ordering on the points is arbitrary; however, for the third proof we need the ordering.

Proof 1: $n = R_a(k)$. We have any n points in $\mathbb{R}^3$

I DO NOT KNOW HOW TO FINISH THIS PROOF. Need an analog of Klein's theorem in $\mathbb{R}^3$. I am sure that some such theorem is true. Thats why I don't know what a is.

Proof 1': We can use $n = R_a(k,b)$ which is the smallest n such that any 2-coloring of $\binom{[n]}{a}$ has either a RED Homog set of size k or a BLUE homog set of size b. Don't know what a or b are.

Proof 2: $n = R_4(k)$. We have any n points in $\mathbb{R}^3$

$$COL(w,x,y,z) = \begin{cases} RED & \text{if numb of pts in tetra formed by } w,x,y,z \text{ is} \equiv 0 \pmod 3 \\ BLUE & \text{if numb of pts in tetra formed by } w,x,y,z \text{ is} \equiv 1 \pmod 3 \\ GREEN & \text{if numb of pts in tetra formed by } w,x,y,z \text{ is} \equiv 2 \pmod 3 \end{cases} \tag{1}$$

A mod-3 argument works here.

Proof 3: $n = R_a(k)$. We have any n points in $\mathbb{R}^3$

Color sets of a-points based on *orientation*. I do not know what that means or how to finish this proof. ∎

There is a generalization to $\mathbb{R}^d$. There was a debate about if you need to increase the colors or if you use 2 colors for $d \equiv 0 \pmod 2$ and 3 colors for $d \equiv 1 \pmod 2$. I leave you to figure all of that out.

4 Large Ramsey, Those ϕ-functions, and the Busy Beaver Function

Recall the following

Def 4.1 Let $H \subseteq \mathbb{N}$. H is *large* if $|H| > \min(H)$.

Theorem 4.2 *(a-ary Large Ramsey) For every $a, c \in \mathbb{N}$, for every k there exists n such that for every coloring $COL : \binom{\{k,\ldots,n\}}{a} \to [c]$ there exists a homog H that is large. We denote n by $LR(a,k,c)$.*

The function $LR(a,k,c)$ grows very fast. How fast? First we put it in terms of one variable: $LR(x,x,x)$.

We define a sequence of functions to demonstrate.

Def 4.3

1. $\Phi_0(x) = x + 1$

2. $\Phi_1(x) = \Phi_0^{(x)}(x)$. This means we do $\Phi_0(\Phi_0(\cdots))$ x times. $(\cdots(x+1)+1)\cdots) = 2x$.

3. $\Phi_2(x) = \Phi_1^{(x)}(x)$. This is $x2^x$.

4. $\Phi_{n+1}(x) = \Phi_n^{(x)}(x)$.

These functions are all *Primitive Recursive*. The function Φ_n is at the nth level of the Primitive Rec Hierarchy. All primitive recursive functions are bounded by some Φ_n. We now define a function that is NOT Primitive Recursive

$$\Phi_\omega(x) = \Phi_x(x).$$

This function eventually grows faster than any Φ_i and hence is not Primitive Recursive. This function is close to Ackermann's function, the standard example of a non-prim-rec function.

So does $LR(x, x, x)$ grow about as fast as $\Phi_\omega(x)$? No. $LR(x, x, x)$ grows much faster.

We can define

$$\Phi_{\omega+1}(x) = \Phi_\omega^{(x)}(x)$$

We can keep defining $\Phi_{\omega+2}$, $\Phi_{\omega+3}$, and so on- until

$$\Phi_{2\omega}(x) = \Phi_{\omega+x}(x).$$

More generally:

Def 4.4 Let α be a countable ordinal.

1. If $\alpha = \beta + 1$ then

$$\Phi_\alpha(x) = \Phi_\beta^{(x)}(x).$$

2. If α is NOT one more than some other ordinal (like ω and 2ω) then there is a sequence that converges to them $\alpha_1, \alpha_2, \ldots$. Now define

$$\Phi_\alpha(x) = \Phi_{\alpha_x}(x).$$

Let α be the limit of ω, ω^ω, ω^{ω^ω}, ….
$LR(x,x,x)$ grows at around the same rate as Φ_α.

5 Are There Faster Functions Than $LR(x,x,x)$?

The function $LR(x,x,x)$ certainly grows faster. Are there functions that grow faster? The obnoxious answer is

$$LR(x,x,x)+1.$$

One can also construct contrived functions that grow faster. Are there natural functions that grow faster (I will not define natural).

Note that $LR(x,x,x)$ is computable. One could write a program that will, on input x, computer $LR(x,x,x)$. One would not want to and one would not want to run such a program. We define a non-computable function that NO computable function can bound.

Def 5.1 Let $M_1, M_2, \ldots,$ be a list of all Turing Machines (if you do not know what Turing Machines are than it can be a list of all Java Programs).

We give a procedure to compute $BB(x)$, though one of the steps one could not really do.

Run $M_1(0), \ldots, M_x(0)$ until those that are going to halt, halt (we do not know which ones will halt, so this really could not be done). Let t be the max time taken by all those that halt, to halt.

If f is ANY computable function then there exists an x_0 such that

$$(\forall x \geq x_0)[f(x) < BB(x)].$$

Since $LR(x,x,x)$ is computable, $BB(x)$ dominates it.

Is $BB(x)$ natural? Perhaps not since it involves Turing Machines. In that light, $LR(x,x,x)$ may be the fastest growing natural function, or perhaps the fastest growing natural computable function.