One Triangle, Two Triangles

William Gasarch

Lets Party Like Its 2019

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If there are 6 people at a party, either 3 know each other or 3 do not know each other.

We state this in terms of colorings of edges of graphs.

For all 2-coloring of the edges of K_6 *there is a mono* K_3.

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then we have a **RED** K_3.

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I either case we get a mono K_3's.

Trivial Theorem, Non Trivial Extension

For all 2-cols of edges of K_{12} there are 2 mono K_3's

Question Find n such that

1. For all 2-coloring of the edges of K_n there are 2 mono K_3's
2. There exists a 2-coloring of the edges of K_{n-1} that does not have 2 mono K_3's.

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1. For all 2-coloring of the edges of K_6 there are 2 mono K_3's
2. There exists a 2-coloring of the edges of K_5 that does not have 2 mono K_3's.

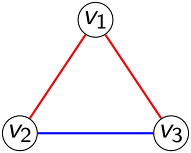
Proof of K_6 Two Triangles Theorem

Theorem For all 2-cols of edges of K_6 there are 2 mono K_3's
Proof Let COL be a 2-coloring of the edges of K_6.
Let R, B, M, be the SET of **RED**, **BLUE**, and **MIXED** triangles.

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \leq 18$, so $|R| + |B| \geq 2$.

A Mixed Triangle Has a Vertex Such That



- (v_2, v_1) is red, (v_2, v_3) is blue. View this as $(v_2, \{v_1, v_3\})$.
- (v_3, v_1) is red, (v_3, v_2) is blue. View this as $(v_3, \{v_1, v_2\})$.

Map ZAN to M

Definition A **Zan** is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \notin \{u, w\}$ and $COL(v, u) \neq COL(v, w)$. ZAN is the set of Zan's.

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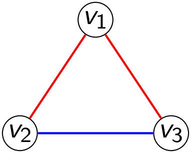
What Zan's map to the triangle:



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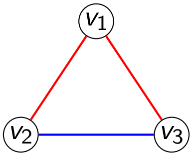

$(v_2, \{v_1, v_3\})$ and $(v_3, \{v_1, v_2\})$.

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So v contributes $\deg_R(v) \times \deg_B(v)$.

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So there are at least 2 Mono Triangles.

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We find an upper bound on $|ZAN|$.

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$$|M| = |ZAN|/2 \leq \frac{(n-1)^2 n}{8}$$

Finishing Up The Proof

Recap

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$$= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$$

Can This Be Improved?

The bound is known to be tight.

An Example of a Ramsey Game

1. The board is a graph on 9 vertices. Known that in any 2-coloring there will be at least 72 mono triangles.
2. Alice and Bob alternate coloring edges. Alice uses **RED**, Bob uses **BLUE**.
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Variants:

1. Whoever gets the most K_3 in their color wins. Could be a tie.
2. Alice wants to get 37 **RED**, Bob just wants to stop her.
3. Either player can use either color and whoever completes the xth mono K_3 wins.
4. There are other variants.

ML

- People in Math have called finding **winnings** strategies for such games **hopeless**.
- The are probably right.
- But ML can help us find **good** strategies.
- Next week Josh will give an ML talk.