1 ADD to the section on SAT

DONE

1.1 Graph Formulas

Bodirsky and Pinsker [1] proved an analog of Schaefer's Theorem for graph formulas. We describe what they did.

Def 1.1 A *Graph Formula* is a Boolean Formula where all of the literals are of the form $E(x, y)$ or $x = y$. A graph formula $\phi(x_1, \ldots, x_n)$ is *satisfiable* if there exists a graph G and a set of vertices of G, $v_1, \ldots, v_n$ such that $\phi(v_1, \ldots, v_n)$ is true in G.

Example 1.2

1. $E(x, y) \wedge E(y, z) \wedge E(z, x)$ is satisfiable. Just take K_3.
2. $$\left(\bigwedge_{1\le i<j<\le 6}(x_i \ne x_j)\right)\bigwedge\left(\bigwedge_{1\le i<j<\le 6}[E(x_i, x_j) \rightarrow E(x_j, x_i)]\right)\bigwedge$$

$$\left(\bigwedge_{1\le i<jk<\le 6}\neg(E(x_i,x_j)\wedge E(x_i,x_k)\wedge E(x_i,x_j))\wedge\neg(\neg E(x_i,x_j)\wedge\neg E(x_i,x_k)\wedge\neg E(x_i,x_j)\right)$$

The first part says that $x_1, \ldots, x_6$ are all different. The second part says that the graph restricted to $x_1, \ldots, x_6$ is symmetric. The third part says that the graph restricted to $x_1, \ldots, x_6$ has neither a clique of size 3 or an independen set of size 3. We leave it to the reader to show that there is no such graph, so this formula is not satisfiable.

Schaefer's Theorem classified types of SAT-problems as being either P or NP-complete. Bodirsky and Pinsker [1] did the same for types of graph formulas.

Let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a set of Boolean formulas. We define a problem Graph-SAT(Ψ).

Problem 1.3 *Graph-SAT(Ψ) Let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a set of Boolean formulas. These are not the input. They are a parameter of the problem. INSTANCE: : A graph formula of the form $\Phi = \phi_1 \wedge \cdots \wedge \phi_L$ where each ϕ_i is one of the ψ_j except that it may use variables other than those used in ψ_j. QUESTION: : Is Φ satisfiable?*

Example 1.4

1. Ψ has the following two formulas:

   - $TRI(x_1, x_2, x_3) = E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_2, x_1)$.
   - $SQUARE(y_1, y_2, y_3, y_4) = E(y_1, y_2) \wedge E(y_2, y_3) \wedge E(y_3, y_4) \wedge E(y_4, y_1)$.

   Consider the following instance:

$$\left(\bigwedge_{1 \le i < j \le 4} x_i \neq x_j \right) \bigwedge TRI(x_1, x_2, x_3) \bigwedge SQUARE(x_1, x_2, x_3, x_4).$$

   This is asking if there is a graph which has four vertices that form a square and three of them also form a triangle. The answer is YES.

   Consider the instance:

$$\left(\bigwedge_{1 \le i < j \le 4} x_i \neq x_j \right) \neg TRI(x_1, x_2, x_3) \bigwedge TRI(x_1, x_3, x_5) \bigwedge SQUARE(x_1, x_{2,3}, x_4).$$

   We leave it to the reader to show that the answer is NO.

The main theorem of Bodirsky and Pinsker [1], which is an an analog of Schaefer's Theorem, is as follows.

Theorem 1.5

1. *For all Ψ the problem Graph-SAT(Ψ) is either* NP*-complete or in* P*.*

2. *The problem of, given Ψ determining of Graph-SAT(Ψ) is* NP*-complete or in* P *is decidable.*

2 MORE TO ADD

Exercise 2.1 Read Erich Friedman's paper [3] on the NP-completeness of the Spiral Galaxies problem. The proof uses a reduction of Circuit SAT. Rewrite the reduction in your own words.

Exercise 2.2 Read Franck Dernoncourt's paper [2] on the NP-completeness of the TruckMania problem. The proof uses a reduction of 3-SAT. Rewrite the reduction in your own words.

Exercise 2.3 Recall the problem ZOP from Section **??**. Show that ZOP is NP-complete by a reduction from 3-SAT.

Exercise 2.4 Consider the following puzzle. The board is a grid. Initially some spaces have a red stone, some have a blue stone, and some are blank. The goal is to remove stones so that for every column: (1) there is at least one stone, and (2) all of the stones in it are the same color. Show that the problem of, given an initial position, can the player win, is NP-complete. Use a reduction from 3-SAT.

Exercise 2.5 Show that the following problem is NP-hard: Given a nondeterministic finite autamoton M, does there exist a string that is not accepted by it?

Exercise 2.6 Show that the following problem is NP-complete: Given a 3CNF formula, is there a satisfying assignment such that a majority of the clauses have all three literals set to T?

References

[1] Manuel Bodirsky and Michael Pinsker. Schaefer's theorem for graphs. *Journal of the Association of Computing Machinery*, 62(3):19:1–19:52, 2015.
`https://arxiv.org/pdf/1011.2894.pdf`.

[2] Franck Dernoncourt. TrackMania is NP-complete, 2014.
https://arxiv.org/pdf/1411.5765v1.pdf/.

[3] Erich Friedman. Spiral galaxies puzzles are NP-complete, 2002.
https://slideplayer.com/slide/254461/.