1 Further Results

1.1 Graph Problems

1. The Planar Multiway Cut Problem: Given a planar graph G with k terminal vertices, find a minimum set of edges whose removal pairwise separates the terminals from each other. k is the parameter. Marx et al. [8] showed that, (1) assuming ETH, for all computable f there is no $f(k) \cdot n^{o(\sqrt{k})}$, time algorithm for this problem and (2) this problem is W[1]-hard.

2. Firefighter problem: Consider the following model of how fires spread. A graph $G = (V, E)$ and a vertex $s \in V$ are given. At time $t = 0$ vertex s ignites. Firefighters then protect one node from being burned. At time $t = 1$ all of the neighbors of s that are not protected ignite. Firefighters then protect one node. At time t all unprotected neighbors of burning vertices ignite, and the firefighters protect one vertex. The process continues until the fire can no longer spread. The problem is to to find a strategy for the firefighters that minimizes the number of burned vertices. This problem is NP-hard. (It is not known to be in NP.) Bazgan et al. [1] showed that if you take the parameter to be either the number of saved vertices, the number of burned vertices, or the number of protected vertices, then the problem is W[1]-hard.

1.2 Restrictions on Graphs

Some graph problems are in FPT if the graphs are restricted. Courcelle [5] and independently Borie et al. [3] showed the following: Let $\mathcal{P}$ be some graph property that is definable in Monadic Second Order Logic, Let $k \in \mathsf{N}$. There is a linear time algorithm for $\mathcal{P}$ restricted to graphs of treewidth $\leq k$. (See Flum & Grohe [6] for a complete treatment of Courcelle's theorem and its applications.) From this theorem the following problems are FPT with the parameter being the tree-width.

1. Given a Boolean Circuit, is it satisfiable?

2. Given a graph G and a number k, is G k-colorable?

3. Given a graph G and a number k, does G gave an Independent set of size k?

4. Given a graph G and a number k, is the crossing number of $G \leq k$? (This one has parameter k + Treewidth.)

These results raised the question of whether bounding the clique width can also be used to put a problem into FPT.

Let f be any computable function. Let t bound the clique number. Assume ETH. Fomin et al. [7] showed that the following problems cannot be solved in time $f(t)n^{o(t)}$.

1. Edge Domination Set: Given a graph $G = (V, E)$ and a $k \in \mathsf{N}$ is there a set $E' \subseteq E$ such that (a) $|E| \leq k$, and (b) every $e \in E$ is either in E' or shares a vertex with some $e' \in E'$.

2. Max-Cut: Given a graph G, find the largest cut of edges such that the graph on these edges form a bipartite graph.

3. Maximum Bisection: Given graph G and integer k, decide if there exists a cut of G into two equally sized vertex sets such that the cut has size at least k.

1.3 Problems From Computational Geometry

Bonnet & Miltzow [2] showed the following.

1. Point Guard Art Gallery: Given a simple polygon $\mathcal{P}$ on n vertices, two points are visible if the line segment between them is in $\mathcal{P}$. Find the minimum set S such that every point in $\mathcal{P}$ is visible from a point in S. Bonnet & Miltzow [2] showed the following: (1) assuming ETH, for any computable function f, this problem has no algorithms in time $f(k)n^{o(k/\log k)}$ (2) with parameter $|S|$, this problem is $W[1]$-hard.

2. The Vertex Guard Art Gallery: The same problem as Point Guard Art Gallery but S is now a subset of $\mathcal{P}$. Bonnet & Miltzow [2] showed the following: (1) assuming ETH, for any computable function f, this problem has no algorithms in time $f(k)n^{o(k/\log k)}$ (2) with parameter $|S|$, this problem is $W[1]$-hard.

3. Hypervolume Indicator: is a measure for the quality of a set of n solutions in R^d. The parameter is d . Bringmann & Friedrich [4] showed the following: (1) assuming ETH the problem has no algorithm in time

$n^{o(d)}$, (2) the problem is $W[1]$-hard (3) there is an average case FPT algorithm.

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