1 Further Reading

1.1 Maximum Feasible Linear System (MAXFLS)

Problem 1.1 Maximum Feasible Linear System (MAXFLS)

INSTANCE: A system of linear equations with coefficient in $\mathbb{Z}$: $A \cdot \vec{x} = \vec{b}$.

QUESTION: A maximum subset of the equations that has a solution over $\mathbb{Q}$.

MAXFLS look easy since, given matrix $A$ and vector $\vec{b}$ one can, in polynomial time, do the following (by Gaussian elimination):

- Determine if there is a solution, and if so then find one, and if not then produce a certificate of infeasibility.

- If there is no solution then find a $\vec{x}$ that achieves $A\vec{x}$ is close to $\vec{b}$. More precisely $\vec{x}$ is such that, $A\vec{x} - \vec{b}$ has the least mean squared error.

Nevertheless, Amaldi and Kann [2] showed the following:

- The natural decision formulation of MAXFLS is NP-hard.

- Many variants and restrictions of it are NP-hard.

- Assume $P \neq NP$. Many variants are hard to approximate. The hardness varies with the variant. Some are in APX but not PTAS, and some are harder to approximate than that.

1.2 MAXCUT

Recall the MAXCUT problem has a straightforward $\frac{1}{2}$-approximation algorithm.

Theorem 1.2

1. (Goemans & Williamson [4]) There is a 0.878...-approximation algorithm for MAXCUT (the number is actually $\frac{2}{\pi} \min_{0 \leq \theta \leq \pi} \left( \frac{\theta}{1 - \cos(\theta)} \right)$).

2. (Hastad [5] building on work of Trevisan et al [10]). Assume $P \neq NP$. Let $\epsilon > 0$. there is no $\frac{10}{17} + \epsilon$-approximation algorithm for MAXCUT. Note that $\frac{10}{17} \sim 0.941$. 

1
Note that if our hardness assumption is $P \neq NP$ then we do not get matching upper and lower bounds. In Chapter 7 we will see that, assuming the Unique Game Conjecture, the algorithm of Goemans & Williamson can be shown to be optimal.

1.3 Closest Vector Problem (CVP)

Def 1.3

1. A lattice $\mathcal{L}$ in $\mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$.

2. Let $p \geq 1$. The $p$-norm of a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is

$$||\vec{x}||_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}.$$ 

Note that $p = 2$ yields the standard Euclidean distance.

3. If $p = \infty$ then

$$||\vec{x}||_p = \max_{1 \leq i \leq n} |x_i|.$$ 

4. The distance between $\vec{x}$ and $\vec{y}$ in norm $p$ is $||\vec{x} - \vec{y}||_p$.

In the next problem let $p \in [1, \infty) \cup \{\infty\}$.

Problem 1.4 Shortest Vector Problem in norm $p$ (SVP$_p$)

INSTANCE: A lattice $L$ specified by a basis.

QUESTION: Output the shortest vector in that basis using the $p$-norm.

Lenstra et al. [8] showed that there is a $2^{n/2}$-approximation to SVP$_2$. They used it to obtain an algorithm to factor polynomials. Schnorr [9] improved this to a $2^{n(log \log n)^2/log n}$-approximation. There are many non-approx results which indicate that the results of the type Lenstra and Schnorr obtained are the best possible. We state two of them and refer readers to the introduction of Khot’s paper [6] for earlier results.

Theorem 1.5

1. (Khot [6]) Assume $NP \not\subset RP$. Let $p \in (1, \infty)$. There is no poly-time constant-approx for SVP.
2. (Khot [6]) Assume \( \mathsf{NP} \not\subseteq \mathsf{RTIME}(2^{\mathsf{polylog}(n)}) \). Let \( p \in (1, \infty) \). Let \( \epsilon > 0 \). There is no poly-time \( 2^{(\log n)^{3/2-\epsilon}} \)-approx for SVP.

3. (Aggarwal et al. [1]) Let \( p \in (1, \infty) - 2\mathbb{Z} \). Assume SETH. \( \mathsf{SVP}_p \) cannot be solved in time \( O(2^{(1-\epsilon)n}) \) for any \( \epsilon > 0 \).

There are similar, but not identical, upper and lower bounds for CVP, the closest vector problem.

1.4 Minimum Bisection

**Problem 1.6 Minimum Bisection**

\textit{INSTANCE: A graph } \( G = (V, E) \) \textit{on an even number of vertices.}

\textit{QUESTION: A partition } \( V = V_1 \cup V_2 \) \textit{such that the number of edges from } \( V_1 \) \textit{to } \( V_2 \) \textit{is minimized.}

**Theorem 1.7**

1. (Feige & Krautghamer [3]) There is a \( O(\log n^2) \)-approximation for Minimum Bisection.

2. (Khot [7]) Let \( \epsilon > 0 \). There exists \( \delta > 0 \) (which depends on \( \epsilon \)) such that the following is true: Assume 3-SAT cannot be solved in \( 2^{O(n^\epsilon)} \) time. Then there is no \( (1 + \delta) \)-approximation for Minimum Bisection.

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