CMSC 858M: Fun with Hardness Proofs
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Chpt. 18 ASPS-Hardness: A Method for Obtaining Cubic Lower Bounds

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1 Overview

The main concept in this chapter is to use the assumption that the All Pairs Shortest Paths (ASPS) problem cannot be solved in subcubic time, $O(n^{3-\epsilon})$, similar to how the 3SUM was used to prove quadratic lower bounds. Note that a chapter reference is broken in the book when referring to the chapter containing 3SUM.

2 APSP Definition

Section 18.1 defines APSP and provides two algorithms for solving the problem in $O(n^3)$ time (Floyd and Warshall, Dijkstra's). The Floyd and Warshall algorithm steps are written clearly since the algorithm is intuitive as is the application of Dijkstra's that follows.

3 Definition of APSP-Hardness

This section is clear, but the acronym ASPS is used on accident instead of APSP.

4 Centrality Measures

This section just quickly defines Radius, Center, Diameter, and Median somewhat clearly. Perhaps a graph could be drawn to show each measure, but this is probably unnecessary.

5 Other Measures

As in the previous section, we just have subcubic problem definitions, nothing unclear.

6 DIAM and PBC Subcubic Equivalence

This section shows a subcubic equivalence between DIAM and PBC. I personally think that removing the intuition from inside the algorithm step list would make for a cleaner algorithm. For example, step 2 of $\mathsf{DIAM} \leq_{\mathsf{sc}} \mathsf{PBC}$ is intuition and could probably be moved to before the other two steps (as well as the step 1 without loss of generality statement). In the $\mathsf{PBC} \leq_{\mathsf{sc}} \mathsf{DIAM}$ proof, step 5 could just be an extension of step 4 instead of its own step to indicate its just intuition.

7 NEGTRI

This proof in my opinion was clear and every observation made sense. My only note is that w is used as the weight function and a variable in observation 2. Perhaps use z instead.

8 Connection to SETH and Open Problems

Simple statements of results and open questions, no issues here.

9 Additional 10 Complexity Problems

From Williams and Williams, "Subcubic Equivalences Between Path, Matrix, and Triangle Problems":

1. **Matrix Product Verification -** Verifying the correctness of a matrix product over the (min, +)-semiring is subcubic equivalent. Given matrices A, B, C from R, verify that $A \cdot B = C$.

2. **Replacement paths problem -** given nodes s and t in a weighted direct graph and shortest path P from s to t, compute the length of the shortest simple path that avoids edge e for all $e \in P$. This problem is subcubic equivalent.

3. **Metricity problem -** given an nxn nonnegative matrix A and want to determine whether it defines a metric space on $[n]$, i.e. if A is symmetric, has 0s on diagonal and entries satisfy the triangle inequality. This problem is subcubic hard.

From Amir Abboud, Loukas Georgiadis, Giuseppe F. Italiano, Robert Krauthgamer, "Faster Algorithms for All-Pairs Bounded Min Cuts":

All-Pairs Min Cut - compute a min s-t cut for all pairs of vertices s, t. This problem has a super-cubic lower bound of $n^{\omega-1-o(1)}k^2$ from a reduction from 4-clique (a novel reduction instead of APSP)

From Amir Abboud, Soren Dahlgaard "Popular Conjectures as a Barrier for Dynamic Planar Graph Algorithms":

Dynamic shortest paths problem - preprocess a planar graph G such that insertions/deletions of edges are supported as well aS distance queries between two nodes u, v assuming the graph is planar at all time steps. This problem cannot be solved in time $O(n^{\frac{1}{2}-\epsilon})$ assuming APSP subcubic hypothesis is true.

From Mohika Henzinger, Danupon Nanongkai, SEbastian Krinninger, Thatchaphol Saranurak, "Unifying and Strengthening Hardness for Dynamic Problems via the Online Matrix-Vector Multiplication Conjecture"

1. **Online matrix-vector multiplication Conjecture** - For any constant, $\epsilon > 0$, there is no $O(n^{3-\epsilon})$-time algorithm that solves OMv with an error probability of at most 1/3. Many lower bound results are shown from this, I include one example that follows.

2. **Dynamic Subgraph Connectivity** - Determine if two vertices s, t are in the same connected component at any time step while supporting adding and removing nodes of the graph. This dynamic problem has a polynomial preprocessing time, $m^{\alpha-\epsilon}$ update time, $m^{1-\alpha-\epsilon}$ query time lower bound for any $0 \leq \alpha \leq 1$.

From Andrea Lincoln "Monochromatic Triangles, Intermdiate Matrix Products, and Convolutions"

1. **All Edges Monchromatic Triangle problem** - given an n-node graph with edges labelled a color from 1 to n^2, decide for each edge if it belongs to a monocrhomatic triangle, a triangle whose 3 edges have the same color. If this problem is solved in $T(n)$ time then, Unweighted APSP is solved in $O(T(n)\log^n)$ time.

2. **Min-Max Product** - two matrices A, B, the min max is matrix C where $C_{i,j} = \min_k \max(A_{ik}, B_{kj})$. If Min-Max in time $T(n)$, then Unweighted APSP is solved in $O(T(n)\log n)$ time.