CMSC 858M: Fun with Hardness Proofs
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Chpt. 9 Parameterized Complexity

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1 Overview

This chapter discusses problems with a parameter k that in which the hardness is dependent on k as well as an eventual complexity hierarchy of parameterized problems. Note that the title of the chapter is missing an 'e' in "Parameterized."

2 Introduction

This section is straightforward and easily understood as written. The VC_k algorithm is well explained and the definitions are clear.

3 Vertex Cover and Kernelization to Parameterized Reductions

The next three sections are fairly straightforward with mostly just definitions as well as a fairly clear proof of VC_k algorithm with a kernelization.

4 Complexity Class W[1]

I had trouble following Theorem 9.6.2 proof that K-STEP-NTM reduces to IS, perhaps a simple example of a cell with a clique might be possible to explain better with a detailed caption? As of now, it is hard to follow all the TM notation as its one of the only proofs in the book that really get into TM notations.

5 CLIQ and IS restricted to Regular Graphs is W[1]-Complete

This section has many reductions for the W[1] class, however most are clear. Perhaps graph on top of pg. 188 could be rewritten from a hand-sketch to a proper drawing.

6 Circuit SAT and Weighted Circuit SAT to Lower Bounds no Approximations Via W[1]-Hardness

This section explores the hierarchy further, and in my opinion is not too hard to follow along as there is mostly just a discussion rather than proofs. On page 195, in Definition 9.8.2, instead of saying just "The following are known" maybe add some sort of "Observation" header or something for the list of simple results about the W hierarchy.

7 Additional 10 Complexity Problems

From Daniel Marx, "A Tight Lower Bound for Planar Multiway Cut with Fixed Number of Terminals":

Planar Multiway Cut - Given a planar graph with k terminal vertices, find a minimum set of edges whose removal pairwise separates the terminals from each other. Assuming ETH, there is no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm for this problem and this problem is W[1]-hard.

From Jianer Chen, Yang Liu, Songjian Lu, Barry O'Sullivan, Igor Razgon "A fixed-parameter algorithm for the directed feedback vertex set problem":

Feedback Vertex Set - Find a set of vertices whose removal leaves a graph without cycles. This problem was open for awhile to determine if it is fixed parameter tractable, and this paper proves that the undirected and directed case is indeed fixed parameter tractable.

From Cristina Bazgan, Morgan Chopin, Marek Cygan, Michael R. Fellows, Fedor V. Fomin, Erik Jan van Leeuwen, "Parameterized Complexity of Firefighting":

Firefighter problem - Place firefighters on the vertices of a graph to prevent a fire with known starting point from lighting up an entire graph. At each time step, a firefighter can be placed on an unburned vertex to permanently protect it while the fire spreads to neighboring vertices. This problem is $W[1]$-hard parameterized by number of saved, burned, and protected vertices.

From Edouard Bonnet and Tillmann Miltzow, "Parameterized Hardness of Art Gallery Problems":

1. **Point Guard Art Gallery** - Given a simple polygon $\mathcal{P}$ on n vertices, two points are visible if the line segment betweent hem is in $\mathcal{P}$. Find

the minimum set S such that every point in $\mathcal{P}$ is visible from a point in S. From ETH, this problem has no algorithms in time $f(k)n^{o(k/\log k}$ for $k = |S|$ and is $W[1]$-hard.

2. **The Vertex Guard Art Gallery** - The same problem as Point Guard Art Gallery but S is now a subset of $\mathcal{P}$. This problem also has no algorithms in time $f(k)n^{o(k/\log k}$ for $k = |S|$ and is $W[1]$-hard.

From Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, and Saket Saurabh, "Algorithmic Lower Bounds for Problems Paraemterized by Clique Width"

1. **Red-Blue Capacitated Dominating Set (Red-Blue CDS)** - given integer k and a bipartite graph with red and blue vertices where red vertices r each have an integer capacity c_r. Is there a set of at most k red vertices that dominates the set of blue vertices and no red vertex in the set has to dominate more vertices than its capacity? Red-Blue CDS is an intermediate problem for the problem that follows, and it is shown that by ETH, there is no $f(t)n^{o(t)}$ algorithm for Red-Blue CDS where t is the clique width, a new parameter akin to treewidth.

2. **Max-Cut** - Given a graph G, find the largest cut of edges such that the graph on these edges form a bipartite graph. From a reduction from Red-Blue CDS, by ETH there is no $f(t)n^{o(t)}$ algorithm for Max Cut where t is the clique width of G.

3. **Maximum Bisection** - Given graph G and integer k, decide if there exists a cut of G into two equally sized vertex sets such that the cut has size at least k. From Max-Cut, by ETH there is no $f(t)^{o(t)}$ time algorithm for Maximum Bisection where t is the clique-width.

From Karl Bringmann and Tobias Friedrich, "Parameterized Average-Case Complexity of the Hypervolume Indicator":

Hypervolume Indicator - is a measure for the quality of a set of n solutions in $\mathbb{R}^d$, and this paper shows that on average there is an algorithm that is fixed parameter tractable but in worst case the problem is $W[1]$-hard.