CMSC 858M: Algorithmic Lower Bounds: Fun with Hardness Proofs
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1 Orthogonal vectors

An important problem in the study of quadratic lower bound is the *orthogonal vectors* problem which we formally define below.

Definition 1 (Orthogonal Vectors (OV)) *: Given* n *vectors in* $\{0, 1\}^d$ *where* $d = O(\log n)$*, are there two vectors with inner product zero?*

A naive algorithm of the above problem solves in time $\mathcal{O}(n^2 d)$ by trying all possible pairs. The best known algorithm takes time $\mathcal{O}(n^{2-\Omega(\frac{1}{\log(\frac{d}{n})})})$ [1]. There are no known algorithms for the problem in *truely subquadratic* time $n^{2-\epsilon}$. The Orthogonal Vectors Conjecture (OVC) states that this is not possible.

Definition 2 (Orthogonal Vectors Conjecture (OVC)[9]) *For every* $\epsilon > 0$*, there is a* $c \geq 1$ *such that OV cannot be solved in* $n^{2-\epsilon}$ *time on instances with* $d = c\log(n)$*.*

It is known that OVC is implied by the popular Strong Exponential Time Hypothesis (SETH) [9].

As noted in [7], the OVC conjecture can be used to show hardness for a variety of problems including Edit Distance [2], Frechet Distance [4], Regular Expression Matching [3], approximating the diameter of a graph [8] and Curve Simplification [5].

OVC can be shown to hold for several restricted computational models. In particular, [7] show that:

1. OV has branching complexity $\tilde{\Theta}(n \cdot \min(n, 2^d))$ for all sufficiently large n, d.
2. OV has Boolean formula complexity $\tilde{\Theta}(n \cdot \min(n, 2^d))$ over all complete bases of $O(1)$ fan-in.

3. OV has requires $\tilde{\Theta}(n \cdot \min(n, 2^d))$ wires, in formulas comprised of gates computing arbitrary symmetric functions of unbounded fan-in.

While the above results show that many problems are subquadratic assuming OV is subquadratic, this does not necessarilly imply that they are *equivalent* to OV. As such, [6] study equivalences between OV and different problems. They show that following problems are all equivalent, in that solving one of them in truly subquadratic time would imply a subquadratic solution for others.

1. OV (Definition 1)
2. (Min-IP): Finding a red-blue pair of vectors with minimum inner product, among n blue vectors and n red vectors in $\{0, 1\}^d$.
3. For a constant $p \in [1, 2]$ and $d = n^{o(1)}$, Approximating the ℓ_p-closest red-blue pair among n red points and n blue points in $\mathbb{R}^d$.

An equivalence can also be established for variations of Min-IP where instead of the minimum, the goal is to find the maximum inner product (Max-IP), or find an inner product that equals a given integer (Exact-ip). Constant Approximations of Min-IP (or Max-IP) are also equivalent.
Additionally, with the extra restriction that the set of blue and red points coincide (i.e., there is only one set), Approximating the ℓ_p-furthest point is also equivalent to OV.

References

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