1 The Unique Games Conjecture

Recall that in our definitions of GAP-MAXR the promise is that either (1) there is a label covering with one vertex per $A_i$ and $B_j$ which covers all superedges, or (2) every such label covering covers at most an $\epsilon$ fraction of the superedges. What if we relaxed the promise of part (1)? Consider the following gap problem.

**Def 1.1** $\epsilon$-2-sided-GAP-MAXR.

**INSTANCE:** A bipartite $G = (A, B, E)$ that has the vertices partitioned as in Definition ??.

**QUESTION:** We only look at label cover which takes exactly one element from each $A_i$ and each $B_j$. We are promised that one of the following occurs.

- There is such a label covering which covers fraction $(1 - \epsilon)$ of the superedges.
- Every such label covering covers at most an $\epsilon$ fraction of the superedges.

The question is to determine which case happens.

Khot [16] made the following conjecture.

**Conjecture 1.2** The Unique Games Conjecture (UGC) is that, for all $\epsilon > 0$, $\epsilon$-2-sided-GAP-MAXR is NP-hard. *(The name Unique Games Conjecture comes from another formulation of it.)*

For more on UGC see Khot’s survey [17] and Klarreich exposition [20]. Is the conjecture true? We argue both sides.

**Argument for UGC**

1. UGC has great explanatory power. There are many examples of this. We give one. Consider the Vertex Cover Problem (VC).

- There is a poly time 2-approximation for VC (so returns twice the min number of vertices needed).
- The 2-approximation result is very old. Despite many attempts to improve it it stays stubbornly at 2-approx.
• Dinur and Safra [7] showed that, assuming $P \neq NP$, or all $\epsilon > 0$, VC has no poly time $1.360 - \epsilon$-approximation.

• Khot and Regev [19] showed that, assuming UGC, or all $\epsilon > 0$, VC has no poly time $2 - \epsilon$-approximation.

We note that the proof of the upper bound of 2, and the proof of the lower bound of $2 - \epsilon$, have nothing to do with each other.

2. Khot et al. [18] proved a weaker version, called the 2-2 games conjectures. See also the exposition by Klarreich [21].

Argument for UGC

1. It is possible we will obtain that explanatory power from the assumption $P \neq NP$.

2. Arora et al. [3] obtained a subexponential algorithm for $\epsilon$-sided-GAP-MAXR is NP-hard. Note that the algorithm is not polynomial and has not been improved on since 2010.

Unlike P vs NP and many other conjectures, the community is truly split on this conjecture.

References


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