Planar 3DM is NP-Complete

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Problem:

Given: 3 disjoint sets $R, B, Y$ with equal cardinality $q$ in each set, and a set $T$ made of triples from $R \times B \times Y$ (i.e. $(\forall t \in T)[t \in R \times B \times Y]$)

Determine whether there is a subset of $q$ triples which contain all of the elements of $R \cup B \cup Y$.

An instance of Planar 3DM is a bipartite graph that contains a vertex for each element of $R, B,$ and $Y$, and each triple in $T$. There is an edge connecting a triple to an element only if the element is a member of the triple.

We will say an instance is planar if this graph $G$ is planar.

To prove the NP-Completeness of Planar 3DM, we will do the following:

1. Prove Planar 1-3SAT is NP-C
2. Prove Planar X3C is NP-C
3. Reduce Planar X3C to Planar 3DM to prove Planar 3DM’s NP-Completeness

Planar 1-3SAT

Definition:

Planar 1-3SAT: Given a 3CNF formula $\phi$, is there a satisfying assignment where every clause has exactly one literal set to TRUE?

Theorem: Planar 1-3SAT is NP-C

To prove this, we will provide a reduction:

Planar 3SAT $\leq$ Planar 1-3SAT

Let $\phi = \{L_1, L_2, L_3\}$ (see figure 1).
We want to form $\phi'$ (see figure 2).
To do so, we replace every clause in $\phi$ with:

$$(L_1 \lor a \lor b) \land (\overline{L_2} \lor a \lor c) \land (\overline{L_3} \lor b \lor d).$$

$\phi'$ (figure 2) is planar, therefore Planar 1-3SAT is NP-C.
**Planar X3C**

**Definition:** X3C:

Given the set \( \{0, \ldots, n\} \) where \( n \equiv 0 \pmod{3} \), and sets \( E_1, \ldots, E_m \) of 3 subsets of \( \{0, \ldots, n\} \), does some set of \( \frac{n}{3} \) of the \( E_i \)'s cover all of \( \{0, \ldots, n\} \), without overlapping?

An instance of X3C is a bipartite graph \( G = (V, E) \) such that:
\( V = \{E_1, E_2, \ldots, E_m\} \), and there is an edge between \( E_i \) and \( j \) if \( j \in E_i \).

**Definition:** Planar X3C: An instance of X3C whose graph is planar.

**Theorem:** Planar X3C is NP-Complete.

We will prove this with a reduction: Planar 1-3SAT \( \leq \) Planar X3C.

First consider this diagram:

![Diagram showing set membership](image)

Let a variable \( b \) in the 1-3SAT instance be represented by a cycle of sets. If \( b \) occurs \( r \) times in the instance, then the cycle has \( 2r \) sets with each pair of sets sharing an element. For the case that \( r = 3 \), see Figure 3 below.
Any system this produces will form an exact cover if and only if the external elements of the cycle are alternately covered by sets of the cycle, with alternate external elements covered by sets not in the cycle.

A successive pair of external elements will represent the appearance of $b$ in a clause of the 1-3SAT instance. The two possible alternatives of $b$ being covered internally by the cycle or by sets external to the cycle will represent $b$ being TRUE or FALSE.

Augment the cycle with $r$ additional sets and $2r$ elements by adding a 3-set to one of the external elements in each pair. See Figures 4 and 5 below for this illustration.
The three elements of $b$ will be denoted in the figures as a connector. Either all three, or none of the connected elements will be covered by the sets of the augmented $b$ cycle.

We must verify if negation is handled correctly. Figure 6 below represents a clause in the 1-3SAT instance. A group of 3 external elements is called a *terminal*:
To complete the construction of an X3C instance, identify the three connector elements for \( b \) in \( L_1 \) with one of the terminals of \( L_1 \).

Now, we have to verify that there is an exact cover of this \( L_1 \) configuration.

In this configuration, the 3 internal elements each appear in 3 of the 9 sets.

Thus, 3 of the sets are used and 9 of the 12 elements will be covered internally, and one terminal will be left uncovered.

By using symmetry in Figure 6, we can verify that, if a terminal is covered externally, the remaining elements will be covered internally. Thus, there is an exact cover by 3-sets for this planar X3C instance i.f.f. there is a satisfying truth assignment for the planar 1-3SAT instance.
This establishes the NP-Completeness of Planar X3C.

**Planar 3DM**

**Theorem:** Planar 3DM is NP-Complete.

We will prove this using a reduction Planar X3C $\leq$ Planar 3DM.

We will do this by modifying the X3C instance to show that the elements can be colored red (R), blue (B), or yellow (Y), such that each 3-set is incident with one element of each color.

The cycles in Figure 3 have a coloring such that:

(i) All external elements are B (colored blue)

(ii) Internal elements are alternately colored R and Y

This is shown below in Figure 3a:

![Figure 3a: with colorings](image-url)
The connector elements can be colored so that the 3 elements are colored differently.

The B element is the fixed connector element, but R, Y elements have freedom regarding which of \{R, Y\} they are colored.

The clause in Figure 6 has a 3-coloring in which the three terminals have coloring, from left to right,

(i) RBY
(ii) BYR
(iii) YRB

An example of scenario (i) is illustrated below in Figure 6a:

![Figure 6a: Colored example of Figure 6](image)

The three internal elements each receive a different coloring.

In order to match the connector elements with the terminals, we
would need to augment the variable cycles if the fixed connector element needs to be colored R or Y. See Figure 7 for this configuration.

Using this component, we can match all terminals by changing the coloring if necessary.

This establishes the NP-Completeness of Planar 3DM.