Change of a Dollar

By William Gasarch

1. a_n is the number of ways to make change of n cents using pennies. Clearly $(\forall n)[a_n = 1]$. We define $a_0 = 1$ and, $(\forall n \leq -1)[a_n = 0]$.

2. b_n is the number of ways to make change of n cents using pennies and nickels. We take $b_0 = 1$ and $(\forall n \leq -1)[a_n = 0]$. Clearly $(\forall n)[b_n = a_n + b_{n-5}]$.

3. c_n is the number of ways to make change of n cents using pennies, nickels, and dines. We take $c_0 = 1$ and $(\forall n \leq -1)[c_n = 0]$. Clearly $(\forall n)[c_n = b_n + c_{n-10}]$.

4. d_n is the number of ways to make change of n cents using pennies, nickels, dimes, and quarters. We take $d_0 = 1$ and $(\forall n \leq -1)[d_n = 0]$. Clearly $(\forall n)[d_n = c_n + d_{n-25}]$.

One can easily prove the following by induction on n:

Theorem 0.1 $(\forall n)[b_n = \left\lfloor \frac{n}{5} \right\rfloor + 1]$.

We can now solve the change-for-a-dollar problem by hand. We need to compute d_{100}. We use the exact formula for b_n and the recurrences for c_n and d_n.

$d_{100} = c_{100} + c_{75} + c_{50} + c_{25} + c_0$

$c_0 = 1$

$c_{25} = b_{25} + b_{15} + b_5 = 6 + 4 + 2 = 12$

$c_{50} = b_{50} + b_{40} + b_{30} + b_{20} + b_{10} + b_0 = 11 + 9 + 7 + 5 + 3 + 1 = 36$

$c_{75} = b_{75} + b_{65} + b_{55} + b_{45} + b_{35} + c_{25} = 16 + 14 + 12 + 10 + 8 + 12 = 72$

$c_{100} = b_{100} + b_{90} + b_{80} + b_{70} + b_{60} + c_{50} = 21 + 19 + 17 + 15 + 13 + 36 = 121$

Hence

$d_{100} = 1 + 12 + 36 + 72 + 121 = 242.$