Lower Bounds on $R(k)$-I

Exposition by William Gasarch

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We compare our LBs to the UB 2^{2k} for convenience.

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a 2-coloring of the edges of $K_{f(k)}$ such that there is no mono K_k.

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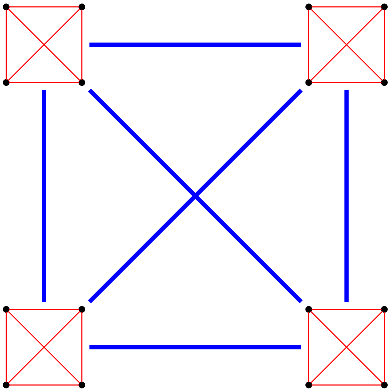
We first give an example, on the next slide.

Example: The $k = 5$ Case



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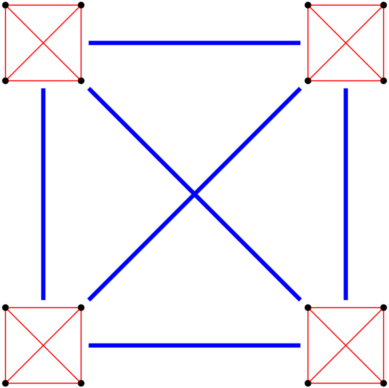

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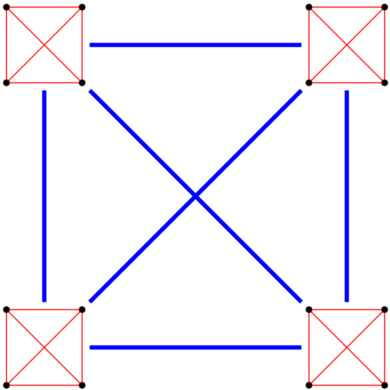

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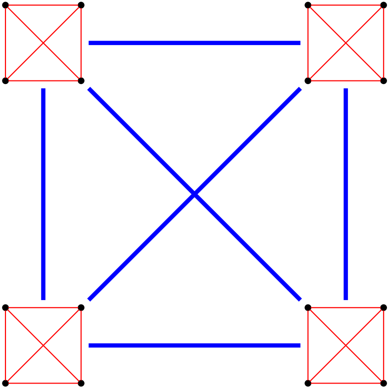

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$4 \times 4 = 16$ vertices. No mono K_5.

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We will do better!