

BILL, RECORD LECTURE!!!!

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Finding Small Dominating Set Via the Prob Method

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Dominating Sets

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We sketch a proof that every graph with min degree d has a dominating set of size $\leq f(n, d)$ where $f(n, d) < n$.

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$$E(|X \cup Y|) = E(|X|) + E(|Y|) \leq np + n(1 - p)^{d+1}.$$

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We need to minimize the following function on the interval $[0, 1]$.

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How good is this? Next Slide.

Table of $d:10-100$

d	$\frac{\ln(d+1)+1}{d+1}$
10	0.3089
20	0.192596
30	0.143032
40	0.114965
50	0.0967025
60	0.0837848
70	0.0741223
80	0.0665981
90	0.0605589
100	0.0555953

Table of $d100-1000$

d	$\frac{\ln(d+1)+1}{d+1}$
100	0.0555953
200	0.0313597
300	0.0222828
400	0.0174413
500	0.0144044
600	0.0123105
700	0.0107739
800	0.00959533
900	0.00866094
1000	0.00790085

Table of $d1000-10000$

d	$\frac{\ln(d+1)+1}{d+1}$
1000	0.00790085
2000	0.00429855
3000	0.00300123
4000	0.00232299
5000	0.0019031
6000	0.00161634
7000	0.00140749
8000	0.00124826
9000	0.00112266
10000	0.00102094

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2. If a graph has min degree ≥ 1000 then there is DS size $\leq 0.008n, \frac{2n}{250}$.
3. If a graph has min degree ≥ 10000 then there is DS size $\leq 0.002n, \frac{n}{500}$.

The Theorem Restated Completely

Thm If $G = (V, E)$ is a graph on n vertices with min degree $\geq d$ then G has a dominating set of size

$$\leq n \left(\frac{\ln(d+1) + 1}{d+1} \right).$$

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Note The statement of the theorem does not mention probability, but the proof uses probability.