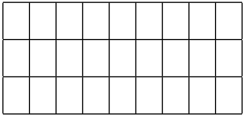
Grid Colorings that Avoid Rectangles

June 16, 2025

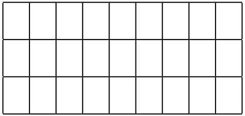
Credit Where Credit is Due

This talk is based on a paper by
Stephen Fenner
William Gasarch
Charles Glover
Semmy Purewal

2-Coloring 3×9

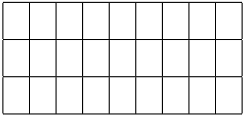


2-Coloring 3×9



Is there a 2-coloring of 3×9 with no mono rectangles?

2-Coloring 3×9



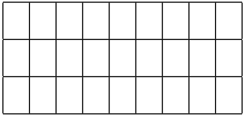
Is there a 2-coloring of 3×9 with no mono rectangles?
What is a mono rectangle? Here is an example:

2-Coloring 3×9

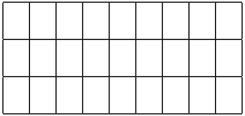
Is there a 2-coloring of 3×9 with no mono rectangles?
What is a mono rectangle? Here is an example:

	R					R		
	R					R		

2-Coloring 3×9: Vote

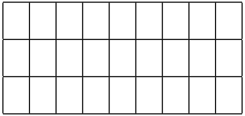


2-Coloring 3×9: Vote



Vote

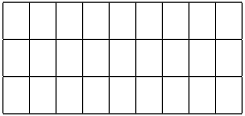
2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.

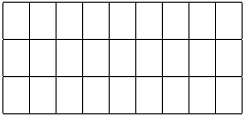
2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.

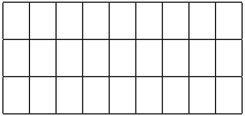
2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.
3. The problem is **UNKNOWN TO SCIENCE**.

2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.
3. The problem is **UNKNOWN TO SCIENCE**.

Answer on the next slide.

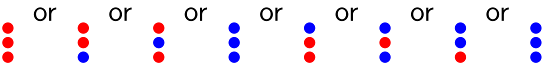
All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

All 2-colorings of 3 $\times$ 9 have a mono rectangle

Given a 2-coloring of 3 $\times$ 9 look at each column.

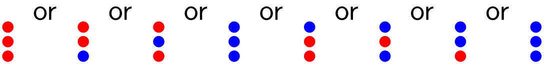
Each column is either



All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

Each column is either

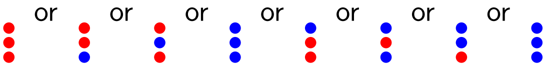


Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

Each column is either



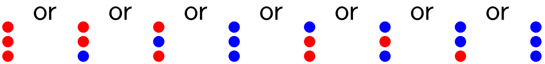
Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

So some column-color appears twice.

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

Each column is either



Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

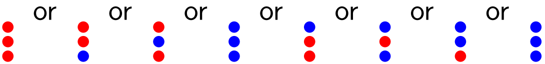
So some column-color appears twice.
Example:

	R					R		
	B					B		
	R					R		

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

Each column is either



Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

So some column-color appears twice.
Example:

	R					**R**		
	B					**B**		
	R					**R**		

Can easily show that the two repeat-columns lead to a mono rectangle.

2-Coloring 3×8, 3×7, ...

Work in groups:

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

Work in groups:

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2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?

2-Coloring 3×8, 3×7, . . .

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?
6. Is there a 2-coloring of 3×3 with no mono rectangles? YES:

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?
6. Is there a 2-coloring of 3×3 with no mono rectangles? YES:

Example:

R	B	R
R	B	B
R	R	B

2-Coloring 3×8, 3×7, . . .

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

   Easily get mono rectangle.

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

   Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

   Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col  OR 

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

   Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col  OR 

   Easily get mono rectangle.

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 
   Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col  OR 
   Easily get mono rectangle.
3. Is there a 2-coloring of 3×6 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

   Easily get mono rectangle.
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   NO: to avoid a repeat col must have col  OR 

   Easily get mono rectangle.
3. Is there a 2-coloring of 3×6 with no mono rectangles?
   YES

2-Coloring 3×8, 3×7, ...

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   NO: to avoid a repeat col must have col 
   Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col  OR 
   Easily get mono rectangle.
3. Is there a 2-coloring of 3×6 with no mono rectangles?
   YES

R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col 

   Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col  OR 

   Easily get mono rectangle.
3. Is there a 2-coloring of 3×6 with no mono rectangles?
   YES

R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

4. Hence there is a 2-coloring of 3×5, 3×4, 3×3 with no mono rectangles.

What Do We Know?

$a \times b$ is *2-colorable* if there is a 2-coloring with no mono rectangles.

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What we know

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What we know

- $2 \times b$ is always 2-colorable

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- $2 \times b$ is always 2-colorable
- $3 \times 3, \ldots, 3 \times 6$ 2-colorable.

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- $2 \times b$ is always 2-colorable
- $3 \times 3, \ldots, 3 \times 6$ 2-colorable.
- $3 \times b$ where $b \geq 7$ NOT 2-colorable.

What Do We Know?

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What we know

- $2 \times b$ is always 2-colorable
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- 4×4, 4×5, 4×6 unknown so far.

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- 5×5, 5×6 unknown so far.

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- 6×6 unknown so far.

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- 5×5, 5×6 unknown so far.
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- 5×5, 5×6 unknown so far.
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Work on the 4×4, 4×5 4×6.

4×6 IS 2-Colorable

R	R	R	B	B	B
R	B	B	R	R	B
B	R	B	R	B	R
B	B	R	B	R	R

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What we know

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- 4×4, 4×5, 4×6 are 2-colorable

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Work on 5×5, 5×6.

5×5 IS NOT 2-Colorable!

Let COL be a 2-coloring of 5×5.

5×5 IS NOT 2-Colorable!

Let COL be a 2-coloring of 5×5.
Some color must occur ≥ 13 times.

Case 1: There is a column with 5 R's

Case 1: There is a column with 5 **R**'s.

$$\begin{matrix} \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \end{matrix}$$

Remaining columns have ≤ 1 **R** so

$$\text{Number of } \mathbf{R}\text{'s } \leq 5+1+1+1+1 = 9 < 13.$$

Case 2: There is a column with 4 R's

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R	$\circ$	$\circ$	$\circ$	$\circ$
R	$\circ$	$\circ$	$\circ$	$\circ$
R	$\circ$	$\circ$	$\circ$	$\circ$
R	$\circ$	$\circ$	$\circ$	$\circ$
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$

Remaining columns have ≤ 2 **R**'s

Number of **R**'s $\leq 4+2+2+2+2 \leq 12 < 13$

Case 3: Max in a column is 3 R's

Case 3: Max in a column is 3 **R**'s.
Case 3a: There are ≤ 2 columns with 3 **R**'s.

$$\text{Number of } \mathbf{R}\text{'s} \leq 3+3+2+2+2 \leq 12 < 13.$$

Case 3b: There are ≥ 3 columns with 3 **R**'s.

R	◦	◦	◦	◦
R	◦	◦	◦	◦
R	**R**	◦	◦	◦
◦	**R**	◦	◦	◦
◦	**R**	◦	◦	◦

Can't put in a third column with 3 **R**'s!

Case 4: Max in a column is $\leq 2R$'s

Case 4: Max in a column is ≤ 2**R**'s.

$$\text{Number of } \mathbf{R}\text{'s} \leq 2+2+2+2+2 \leq 10 < 13.$$

No more cases. We are Done! Q.E.D.

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- $3 \times b$ where $b \geq 7$ NOT 2-colorable.
- 4×4, 4×5, 4×6 are 2-colorable
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Can we say it more succinctly?

Clean Short Statement

Def $n \times m$ contains $a \times b$ if $a \leq n$ and $b \leq m$.

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Thm $n \times m$ is 2-colorable iff $n \times m$ does not contain any of the following grids:

$$\{3 \times 7, 5 \times 5, 7 \times 3\}.$$

3-COLORABILITY

Which Grids are 3-Colorable?

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Clearly $M_{\mathrm{R}} \le M_{\mathrm{N}}$.

Hence, to show $M_{\mathrm{N}} \ge 56$, it suffices to show $M_{\mathrm{R}} > 55$.

Well Known Theorem

The Min of

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occurs when all of the x_is are equal.

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$$\sum_{i=1}^{11} \frac{x_i(x_i-1)}{2}$$

$$\geq 11 \times \frac{41}{11}(\frac{41}{11} - 1)\frac{1}{2} = 55.9090\ldots$$

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Question Is 10×10 3-colorable? See next slide.

10×10 is 3-colorable

Thm 10×10 is 3-colorable.

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

Complete Char of 3-colorability

Techniques and computer work got us this:
Thm The grid $m \times n$ is 3-colorable iff it does not contain any of the following:

$$\{4 \times 19, 5 \times 16, 7 \times 13, 10 \times 11, 11 \times 10, 13 \times 7, 16 \times 5, 19 \times 4\}$$

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Bernd Steinbach and Christian Postoff showed both 17×17 is 4-colorable and they are $289 richer!

Theorem on 4-coloring

$n \times m$ is 4-colorable iff it does not contain any of the following:
$\{5 \times 41, 6 \times 31, 7 \times 29, 9 \times 25, 10 \times 23, 11 \times 22\} \cup$
$\{22 \times 11, 23 \times 10, 25 \times 9, 29 \times 7, 31 \times 6, 41 \times 5\}$.

Questions to Ponder During the Break

Def Let $a, b, c \in \mathbb{N}$. The $a \times b$ is *c-colorable* if there is a coloring of $a \times b$ where there is no set of four points that are the same color, that are the corners of a rectangle.

1) Show that 4×48 is not 3-colorable.

2) Show that 4×18 is 3-colorable.

3) Find a number b such that $5 \times b$ is not 4-colorable.

4) Show that 5×40 is 4-colorable.

5) Is there some number n such that, for all 2-colorings of $n \times n$, there are four points that are the same color that are the corners of a square?