Better Lower Bounds on $R(k)$

Exposition by William Gasarch

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We want much better lower bounds.

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Key This was Erdös 's big breakthrough.

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Prob that a random 2-coloring HAS a homog set is bounded by

$$\frac{\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} \le \frac{\binom{n}{k} \times 2}{2^{\binom{k}{2}}} \le \frac{n^k}{k!2^{k(k-1)/2}}$$

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So if $\frac{n^k}{k!2^{k(k-1)/2}} < 1$ then **there exists** a coloring of the edges with **no homog set of size k**.

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We will work out the algebra of $\frac{n^k}{k!2^{k(k-1)/2}} < 1$ on the next slide; however, the real innovation here is that we show that a coloring exists by showing that the prob that it does not exists is < 1.

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This is **The Probabilistic Method**.

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Want n large. $n = \frac{1}{e\sqrt{2}} k 2^{k/2}$ works.

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Joel Spencer told me he was hoping for a better improvement.

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- I would say that Ramsey Theory was the initial motivation for the Prob Method which is now used for many other things, some of which are practical.