Small Ramsey Numbers

Exposition by William Gasarch

June 16, 2025

Lets Party Like its 1999

The First Theorem In Ramsey Theory

The first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

> *If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

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We define graphs and complete graphs and state this theorem in those terms.

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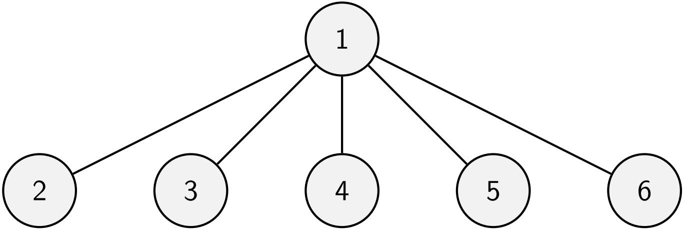

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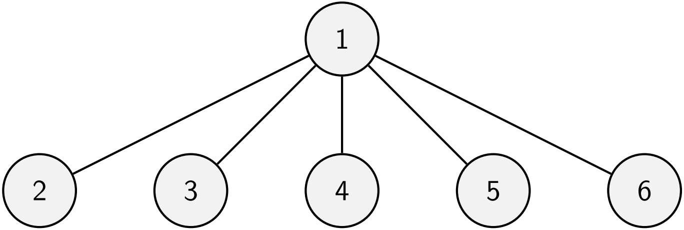

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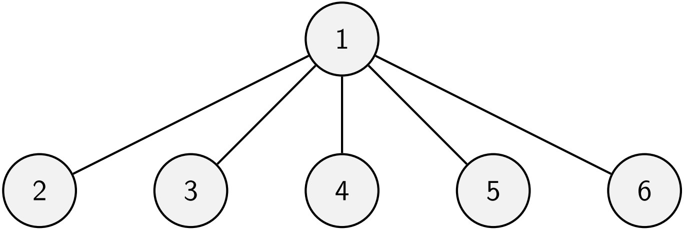

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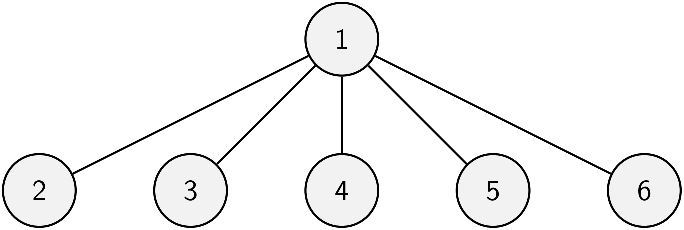

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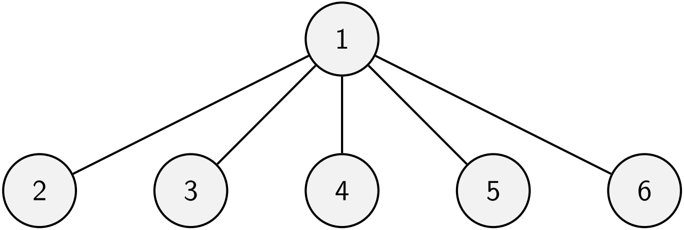

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Def The **degree (deg)** of a vertex is how many edges use it.
In the above graph $\deg(1) = 5$ and
$\deg(2) = \deg(3) = \deg(4) = \deg(5) = \deg(6) = 1$.

Complete Graphs

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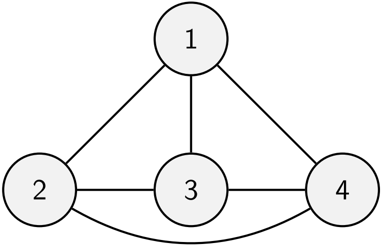

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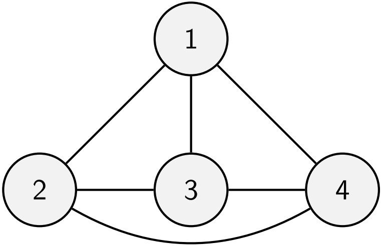

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Note Every vertex of K_n has degree $n - 1$.

More Notation

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Thats a tautology!

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Bill Gasarch and the Red Cliques!

Proof of The First Theorem In Ramsey Theory

The First Theorem, Restated

For every 2-coloring of the edges of K_6 *there is a monochromatic* K_3 *(triangle).*

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We prove this in the next few slides.

The First Theorem in Ramsey Theory

Thm For all 2-colorings of the edges of K_6 there is a mono K_3.

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Given a 2-coloring of the edges of K_6 we look at vertex 1.

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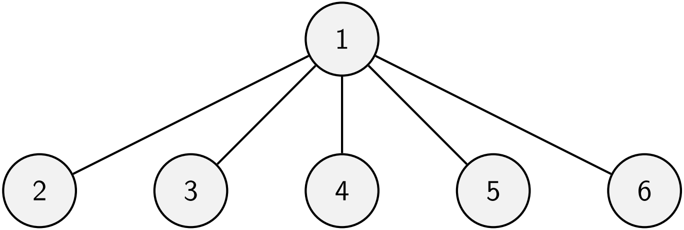

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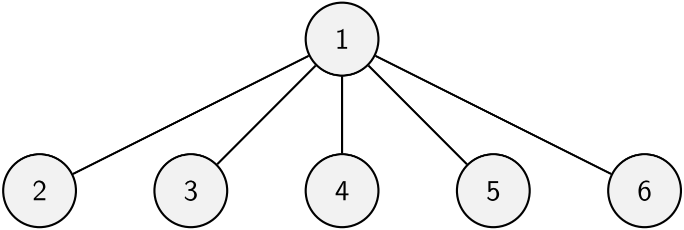

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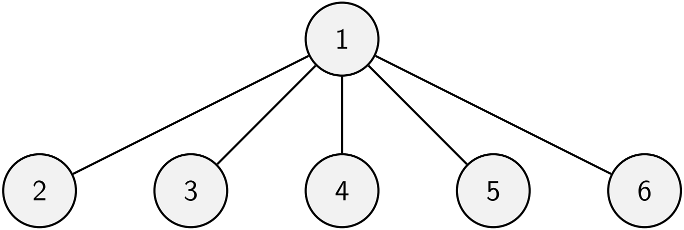

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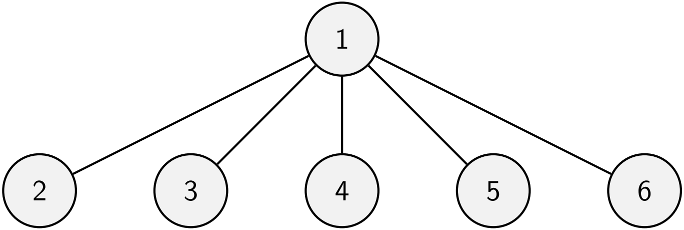

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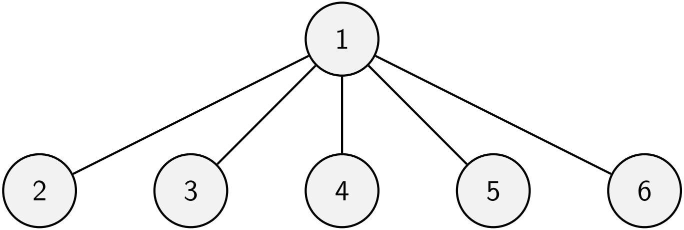
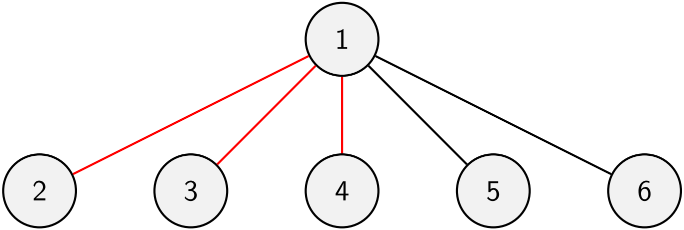
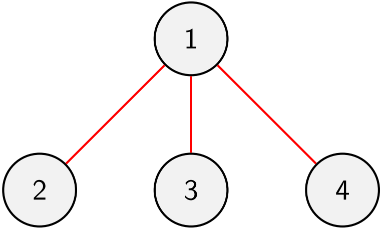


There are 5 edges coming out of vertex 1.
They are 2 colored.
There exists 3 edges from vertex 1 that are the same color.
We can assume $(1,2)$, $(1,3)$, $(1,4)$ are all **RED**.

(1,2), (1,3), (1,4) are RED



We Look Just at Vertices 1,2,3,4



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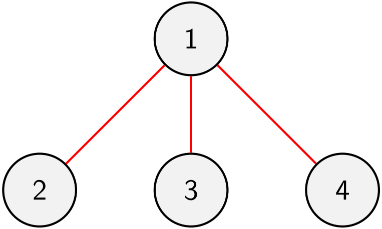

If $(2, 3)$ is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

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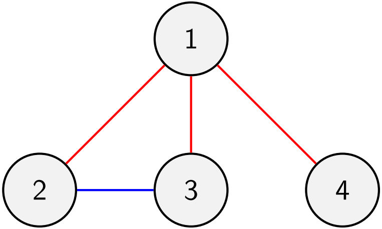

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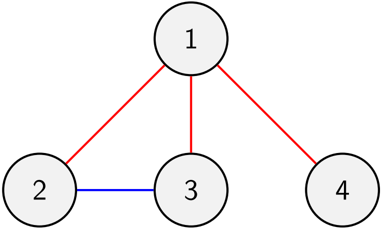

If (3,4) is **RED** then get **RED** triangle. So assume (3,4) is **BLUE**.

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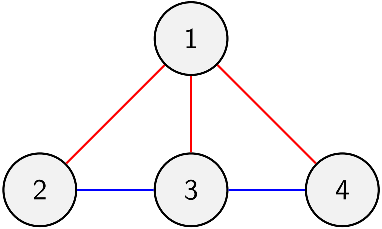

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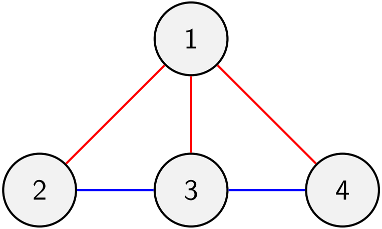

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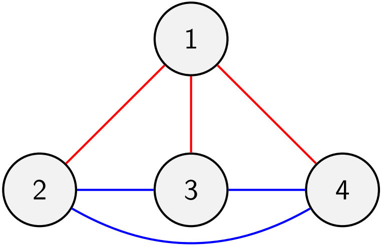

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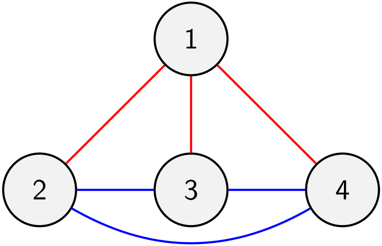

Note that there is a **BLUE** triangle with verts $2, 3, 4$. Done!

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Thm $R(3) = 6$.

Bounds on Asymmetric Ramsey Numbers

Asymmetric Ramsey Numbers

Definition Let $a, b \geq 2$. $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
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Proof left to the reader, but its easy.

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1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.
3. All verts have small **Red** degree and small **Blue** degree.

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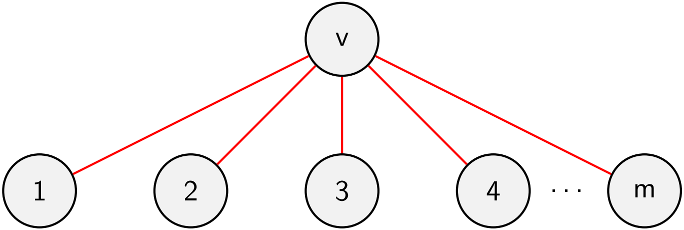

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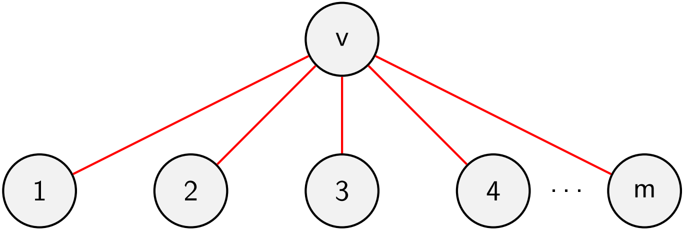

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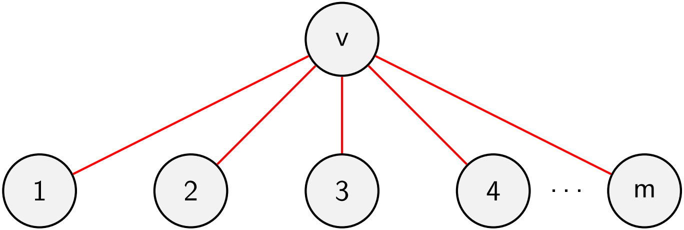

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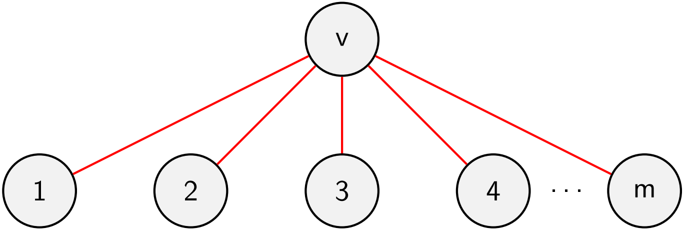

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Case 1.3 Neither. **Impossible** since $m = R(a-1,b)$.

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Case 2 There exists v, $\deg_B(v) \geq R(a, b-1)$.

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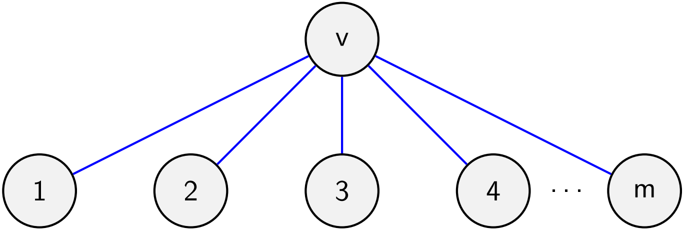

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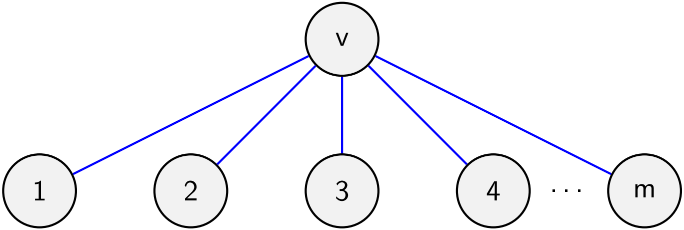

Case 2.1 There is a **Red** K_a in $\{1, \ldots, m\}$. DONE

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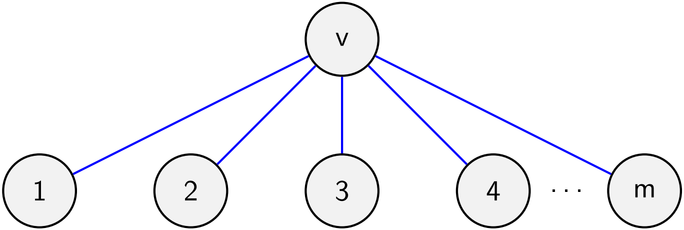

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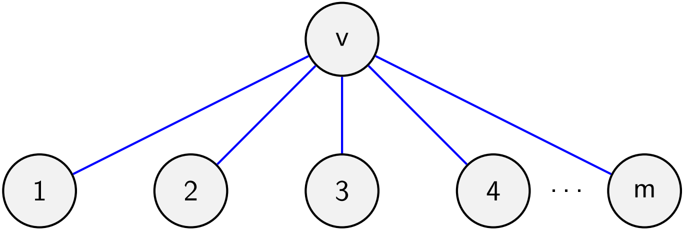

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All Verts: Small Red Deg and Small Blue Deg

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2. for all v, $\deg_B(v) \le R(a,b-1)-1$.

Hence

$$\text{For all } v, \deg(v) \le R(a-1,b)+R(a,b-1)-2 = n-2.$$

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

1. For all v, $\deg_R(v) \le R(a-1,b) - 1$ and
2. for all v, $\deg_B(v) \le R(a,b-1) - 1$.

Hence

$$\text{For all } v, \deg(v) \le R(a-1,b) + R(a,b-1) - 2 = n - 2.$$

Not possible since every vertex of K_n has degree $n-1$.

Lets Compute Bounds on $R(a, b)$

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Can we make some improvements to this?

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Can we make some improvements to this? YES!
That is our next topic!

Better Bounds on Asymmetric Ramsey Numbers

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

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We generalize this on the next slide.

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Handshake Lemma If all pairs of people in a room shake hands, even number of shakes.

Corollary of Handshake Lemma

Impossible to have a graph on an odd number of verts where every vertex is of odd degree.

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Impossible to have a graph on an odd number of verts where every vertex is of odd degree.
And NOW to our improvements on small Ramsey numbers.

$R(3,4) \leq 9$ Case 1

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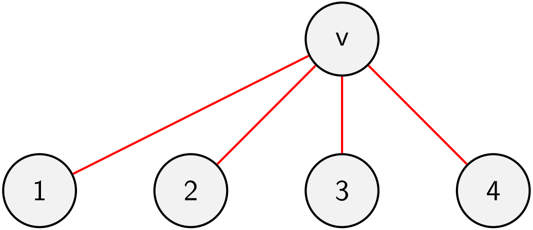

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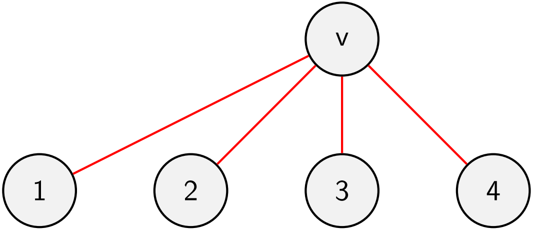

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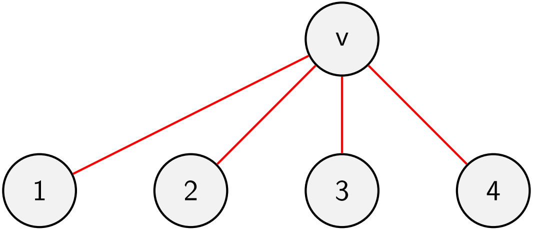

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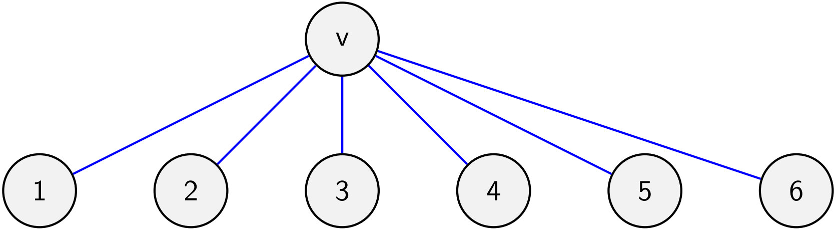
2) If **all** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **BLUE**, have **BLUE** K_4.

$R(3,4) \leq 9$ Case 2

Case 2 There exists v, $\deg_R(v) \leq 2$, so $\deg_B(v) \geq 6$.



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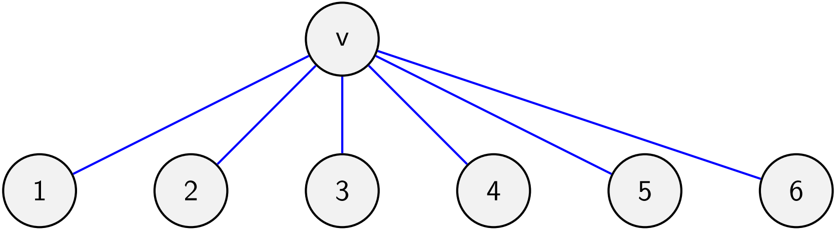

(1) There is a **RED** K_3 in $\{1,2,3,4,5,6\}$. Have **RED** K_3.

$R(3,4) \le 9$ Case 2

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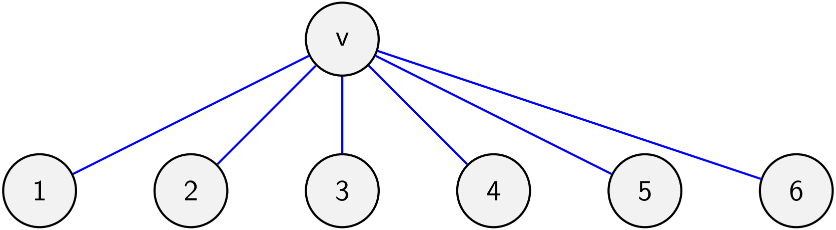

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$R(3,4) \leq 9$ Case 3

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SO the **RED** graph is a graph on 9 verts with all verts of degree 3.
This is impossible!

A Generalization of this Trick

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Theorem $R(a,b) \le$

1. $R(a,b-1) + R(a-1,b)$ always.
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Proof left to the Reader.

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

The Old Bounds and the New Bounds

$R(a, b)$	Old Bound	New Bound
$R(3, 3)$	6	6
$R(3, 4)$	10	9
$R(3, 5)$	15	14
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Are the new bounds tight?

Tightening the Bounds on Asymmetric Ramsey Numbers

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The coloring is on the next slide.

An Interesting Coloring

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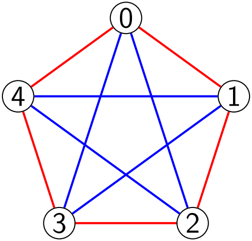

This graph is not arbitrary.

$\mathrm{SQ}_5 = \{x^2 \pmod 5) : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in \mathrm{SQ}_5$ then **RED**.
- If $i - j \notin \mathrm{SQ}_5$ then **BLUE**.

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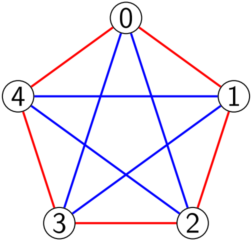

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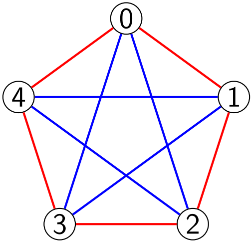

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One can show there are no mono triangles using mod 5.
Upshot $R(3) = 6$ and coloring used interesting math.

$R(4,4) \geq 17$ with Interesting Coloring

$R(4,4) \geq 18$: Need coloring of K_{17} w/o mono K_4.

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Vertices are $\{0,\ldots,16\}$.

Use
$COL(a,b) =$ **RED** if $a-b$ is a square mod 17, **BLUE** OW.

$R(4,4) \geq 17$ with Interesting Coloring

$R(4,4) \geq 18$: Need coloring of K_{17} w/o mono K_4.

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Use
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Upshot $R(4,4) = 18$ and the coloring used interesting math.

$R(3,5) \geq 14$ with Interesting Coloring

$R(3,5) \geq 14$: Need coloring of K_{13} w/o **RED** K_3 or **BLUE** K_5.

$R(3,5) \geq 14$ with Interesting Coloring

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Vertices are $\{0,\ldots,13\}$.

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Upshot $R(3,5) = 14$ and the coloring used interesting math.

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Worse News We do not know any other $R(a,b)$

Summary of Bounds

$R(a,b)$	Old Bound	New Bound	Opt	Int?
$R(3,3)$	6	6	6	Y
$R(3,4)$	10	9	9	Y
$R(3,5)$	15	14	14	Y
$R(3,6)$	21	19	18	Lower-Y
$R(3,7)$	28	27	23	Lower-Y
$R(4,4)$	20	18	18	Y
$R(4,5)$	35	31	25	N
$R(5,5)$	70	62	≤ 46	N

Generalizations

- Instead of 2 colors use c colors
- Instead of coloring pairs-of-vertices color triples-of-vertices.

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   (Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.

2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.

When Will We Know $R(5, 5)$

1. (Quote from Joel Spencer): *Erdos asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of* $R(5, 5)$ *or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for* $R(6, 6)$*. In that case, he believes, we should attempt to destroy the aliens.*

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   $R(5) = 43$, and
   One of the students in the Brin math program will figure it out by the end of the week.

Problems to Ponder over Break

Def Let $a, b, c \geq 2$.
$R(a, b, c)$ is the least n such that, for all 3-colorings of the edges of K_n, either there is a Red a-clique, or there is a Blue b-clique, or there is a Green c-clique.

1) Show that $R(2, b, c) = R(b, c)$.

2) Show that
$R(a, b, c) \leq R(a-1, b, c) + R(a, b-1, c) + R(a, b, c-1)$.

3) Use 1 and 2 to find an upper bound on $R(3, 3, 3)$.

4) Recall that $R(a, b) \leq R(a, b-1) + R(a-1, b)$. From that, derive an upper bound on $R(a, b)$ as a function of a, b.

Problems for Day 2 of Ramsey Theory

Def Let $a, b, c \geq 2$.
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- there is a Red a-clique, or
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1) Show that $R(2, b, c) = R(b, c)$.

2) Show that
$R(a, b, c) \leq R(a-1, b, c) + R(a, b-1, c) + R(a, b, c-1)$.

3) Use Problems 1 and 2 to find an upper bound on $R(3, 3, 3)$.

4) Recall that $R(a, b) \leq R(a, b-1) + R(a-1, b)$. From that, derive an upper bound on $R(a, b)$ as a function of a, b.

5) Fill in the following statement so that its Ramseyian and prove it:
Let $K_{\mathbb{N}}$ be the complete graph on $\mathbb{N}$ (the naturals). For all 2-colorings of the edge of $K_{\mathbb{N}}$ XXX happens.