Bounds on $R(a, b)$

Exposition by William Gasarch

June 16, 2025

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Now lets use it

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$\binom{x}{y}$ is pronounced ***x* choose *y***

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2) Do NOT include Ian! Need to pick b from $a+b-1$, $\binom{a+b-1}{b}$.

So the total number of ways is $\binom{a+b-1}{b-1} + \binom{a+b-1}{b}$.

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Combintorial proofs are better!

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$R(a, b) \leq \binom{a+b}{b}$.

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Proof is mathematically sophisticated- beyond the scope of this weeks mini-class on Ramsey Theory.