Van Der Warden's (VDW) Thm

Exposition by William Gasarch

June 19, 2025

Arithmetic Sequences

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Example Use Board:
For $W = 3, \ldots, 9$ we will see if there is a 2-coloring of $\{1, \ldots, W\}$ with no mono 3-AP.

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$W(3, 2)$ =Hmmm, this is the first non-trivial one.

$W(3,2)$ exists

Do On Board

$W(3,3)$

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How big should the blocks be?

Bill will do it **on the board**.

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- There are ways to formalize the proof; however, they are not enlightening.

What Did We Use to Prove $W(3, c)$?

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Note that we **do not** do
$W(3,2) \implies W(3,3)$.

$W(4,2)$

Bill does at **at blackboard**.

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The numbers are large since they come out of an ω^2 induction.

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Vote: YES, NO, Unknown to Science.

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- Proof is elementary. Could teach to you given more time.
- Bounds still large. About the 11th level.

Deep Math From Search for Better Upper Bounds on VDW Numbers

Exposition by William Gasarch

June 19, 2025

A Man, A Plan, A Canal: Panama!

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We outline a plan for getting better upper bounds on $W(k, c)$.

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It DID succeed! (Oh! Thats a good thing!)

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Examples

1. For all k, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $\frac{1}{k}$.
2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.

A Conjecture, 1936

Conjecture If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.

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Conjecture If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.

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The hope was that the proof of Conj would require a new proof of VDW's Thm that would lead to better bounds.

Roth's Thm, 1952

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- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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- Proof is elementary but strains the use of the word **elementary**.
- The theorem is known as **Szemeredi's Thm**.
- Szemeredi should have won Fields Medal ($15,000) but did not since combinatorics was not seen as deep math.

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  - Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

Known Lower Bounds

1. Easy Use of Prob Method (was on HW) $W(k,2) \ge \sqrt{k}2^{k/2}$ (Easy extension to 3 colors)
2. Very sophisticated use yields $W(k,2) \ge \frac{2^k}{k^\epsilon}$ (Does not extend to 3 colors.)
3. If p is prime then $W(p,2) \ge p(2^p - 1)$. Constructive! (Does not extend to 3 colors.)

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