

ANY TWO-COLORING OF THE PLANE CONTAINS MONOCHROMATIC 3-TERM ARITHMETIC PROGRESSIONS

GABRIEL CURRIER, KENNETH MOORE, CHI HOI YIP

Presenter: Chaewoon Kyoung

Notation

$\mathbb{E}^n$ = n-dimensional Euclidean space

ℓ_m = m collinear points with distance one apart

I.e. m-term arithmetic progression with common difference of 1

Theorem 1.1.

*In any two-coloring of $\mathbb{E}^2$,
there exists a monochromatic congruent copy of ℓ_3 .*

Theorem 1.1.

*In any two-coloring of $\mathbb{E}^2$,
there exists a monochromatic congruent copy of ℓ_3 .*

By scaling, there exists a monochromatic 3-term arithmetic progression
with any common difference.

Theorem 1.2. (Erdős et. al.)

Let $n \geq 2$. A given 2-coloring of $\mathbb{E}^n$ admits a monochromatic (a, b, c) triangle if and only if it admits a monochromatic equilateral triangle of side a, b , or c .

Theorem 1.2. (Erdős et. al.)

Let $n \geq 2$. A given 2-coloring of $\mathbb{E}^n$ admits a monochromatic (a, b, c) triangle if and only if it admits a monochromatic equilateral triangle of side a, b , or c .

An L3 is a $(1, 1, 2)$ monochromatic (degenerate) triangle.

Thus, if there is no L3 in a given coloring of $\mathbb{E}^n$,
there are no monochromatic triangle of side length 1 or 2 in the same coloring

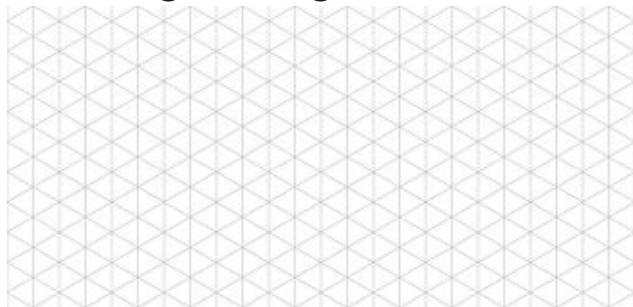
Lemmas we will use

Theorem 1.2. (Erdős et. al.) **Will Not Prove**

Lemma 2.1. If E^2 is 2-colored without a monochromatic L_3 , any unit eq. triangle colored R-R-B has B centroid **Will Prove**

Lemma 2.2. If E^2 is 2-colored without a monochromatic L_3 , a $\frac{1}{\sqrt{3}}$ scaled

Hexagonal grid has one valid coloring **Will Prove**



Proof for Lemma 2.1:

If E^2 is 2-colored without a monochromatic L_3 , any unit eq. triangle colored R-R-B has B centroid

For the Sake of Contradiction...

Assume there is no monochromatic L_3 in an arbitrary 2-coloring of E^2 .

Construct a point set that includes many L_3 s:

$$[a,b,c,d] := \left(\frac{a\sqrt{3} + b\sqrt{11}}{12}, \frac{c + d\sqrt{33}}{12} \right),$$

Let $X = \{ [a,b,c,d] : a,b,c,d \in \mathbb{Z} \}$

Assume there is a 2-coloring of $\mathbb{E}^2$ without a mono L_3 .

For the sake of contradiction,

assume that a unit equilateral triangle of color R-B-B with the 3 points and its centroid in X has a centroid of color R.

Set of 56 points that admits no possible coloring for counterclaim

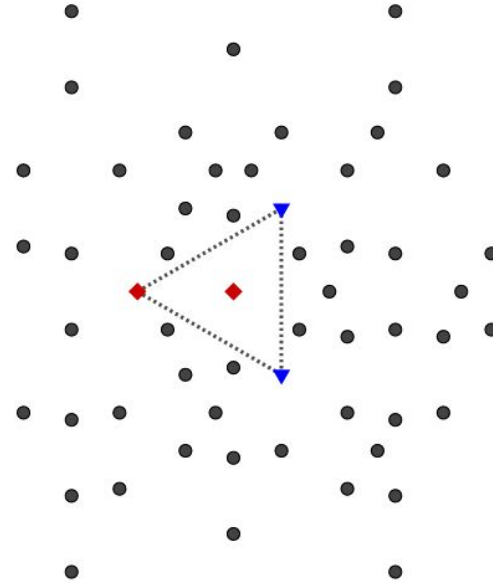


FIGURE 1. A set of 56 points admitting no coloring compatible with the initial coloring of the unit triangle and its center

A subset of the 56 point set

$$\begin{array}{ll} p_1 = [-4, 0, 0, 0], & p_2 = [0, 0, 0, 0], \\ q_1 = [-1, -3, 3, -1], & q_2 = [-1, -3, -3, 1], \\ q_4 = [-3, -3, -3, -1], & q_5 = [-3, -3, 3, 1]. \end{array} \quad \begin{array}{ll} p_3 = [2, 0, -6, 0], & p_4 = [2, 0, 6, 0], \\ q_3 = [2, 0, 0, 2], & q'_3 = [2, 0, 0, -2], \end{array}$$

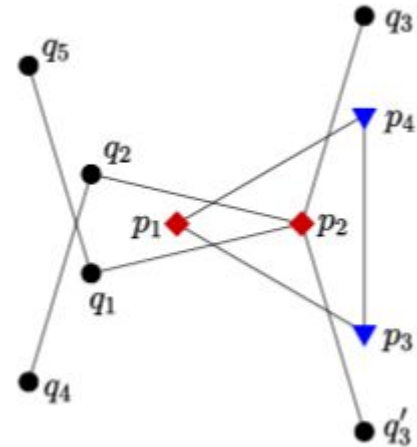


Figure 1: 56 point contradiction

$[-4, 0, 0, 0] p_1, [0, 0, 0, 0] p_2, [2, 0, -6, 0] p_3, [2, 0, 6, 0] p_4, [-1, -3, 3, -1] q_1,$
 $[-1, -3, -3, 1] q_2, [2, 0, 0, 2] q_3, [2, 0, 0, -2] q'_3, v[-3, -3, -3, -1] q_4, [-3, -3, 3, 1] q_5,$
 $[-1, -3, -3, -3], [-1, 3, 3, 1], [-1, 3, -9, 1], [3, 3, -3, -1], [-2, 0, -6, 0], [1, 3, -9, -1],$
 $[-2, 0, 0, -2], [1, -3, -3, -1], [4, 0, 0, 0], [-1, 3, 9, -1], [0, 0, -6, -2], [-1, -3, 9, 1],$
 $[1, 3, -15, 1], [1, -3, 3, 1], [1, 3, -3, 1], [-2, 0, 6, 0], [-3, -3, 9, -1], [1, 3, -3, -3],$
 $[6, 0, 0, 2], [5, -3, -3, -1], [-1, -3, 3, 3], [3, 3, -9, 1], [-5, 3, 3, 1], [6, 0, 0, -2],$
 $[-2, 6, 0, 0], [1, 3, 3, -1], [5, -3, 3, 1], [-1, -3, -15, 1], [0, 0, 6, 2], [3, -3, -3, 1],$
 $[0, 0, 6, -2], [5, 3, -3, 1], [-1, 3, -3, -1], [0, 0, -6, 2], [0, 0, -12, 0], [-3, 3, 3, -1],$
 $[-1, -3, -9, -1], [3, -3, 3, -1], [-2, 0, 0, 2], [-1, 3, 3, -3], [-3, 3, -3, 1], [3, 3, 3, 1],$
 $[5, 3, 3, -1], [1, -3, 3, -3], [1, 3, 9, 1], [1, 3, 3, 3].$

A subset of the 56 point set

$$\begin{array}{ll} p_1 = [-4, 0, 0, 0], & p_2 = [0, 0, 0, 0], \\ q_1 = [-1, -3, 3, -1], & q_2 = [-1, -3, -3, 1], \\ q_4 = [-3, -3, -3, -1], & q_5 = [-3, -3, 3, 1]. \end{array} \quad \begin{array}{ll} p_3 = [2, 0, -6, 0], & p_4 = [2, 0, 6, 0], \\ q_3 = [2, 0, 0, 2], & q'_3 = [2, 0, 0, -2], \end{array}$$

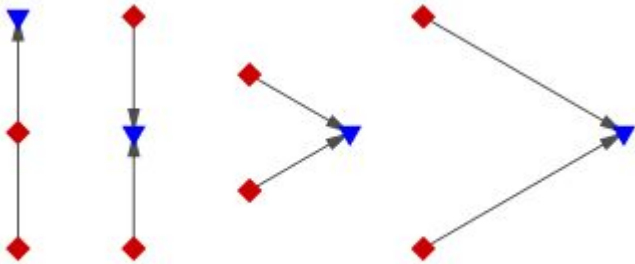
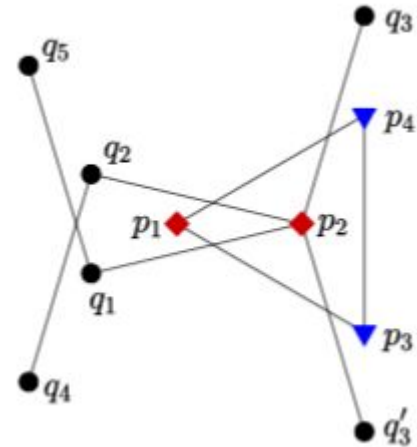
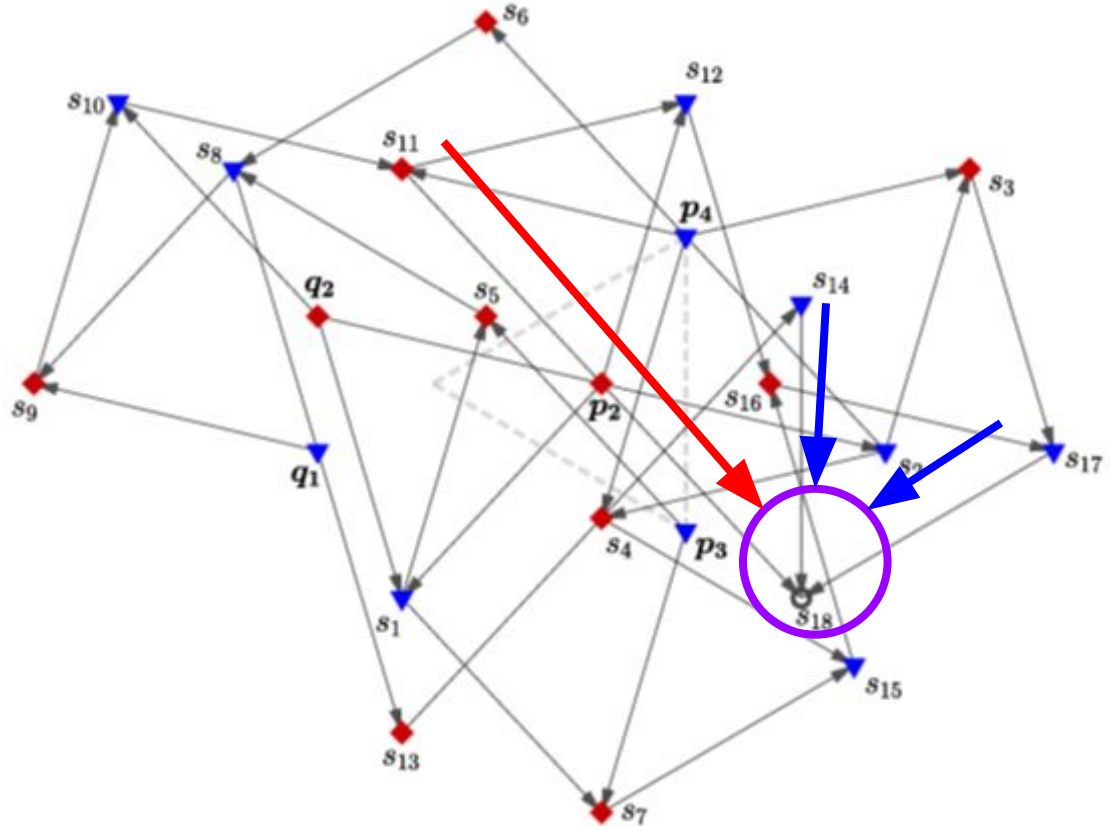


FIGURE 3. The color implication steps



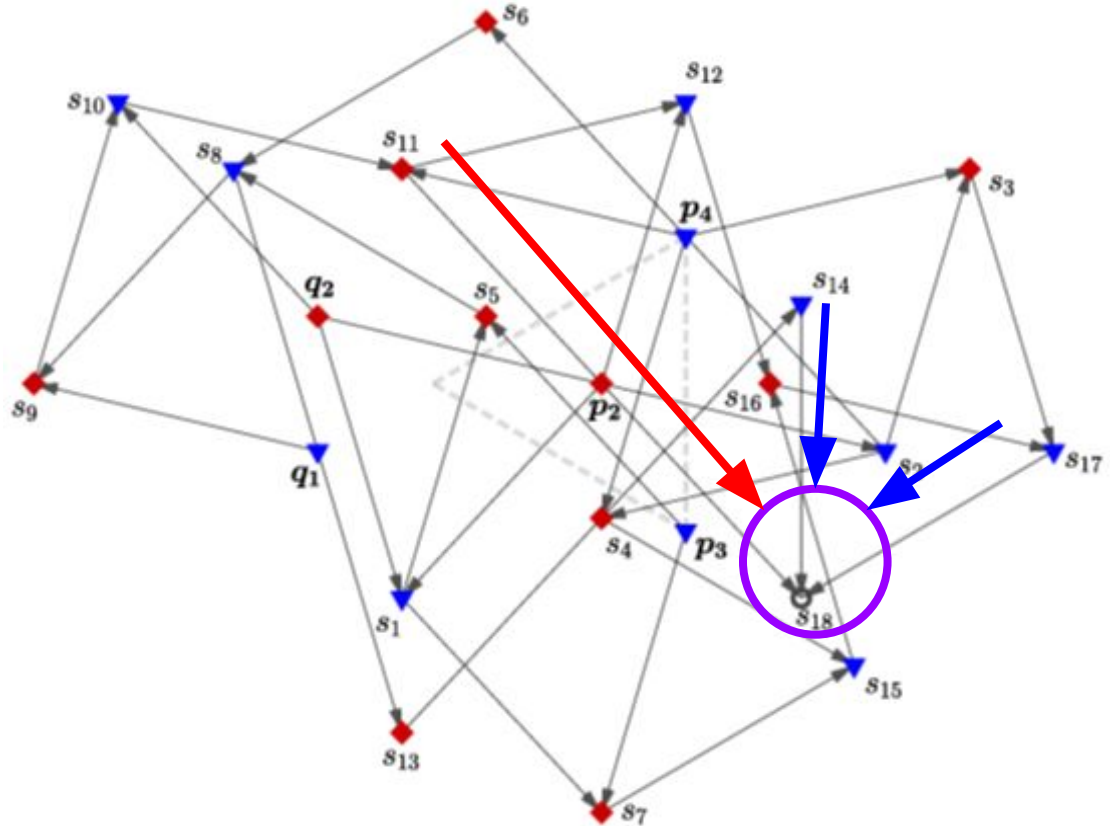
Casework on q1 and q2

q1 R and q2 B results in:



Casework on q_1 and q_2

q_1 R and q_2 B results in:



End of proof for Lemma 2.1.

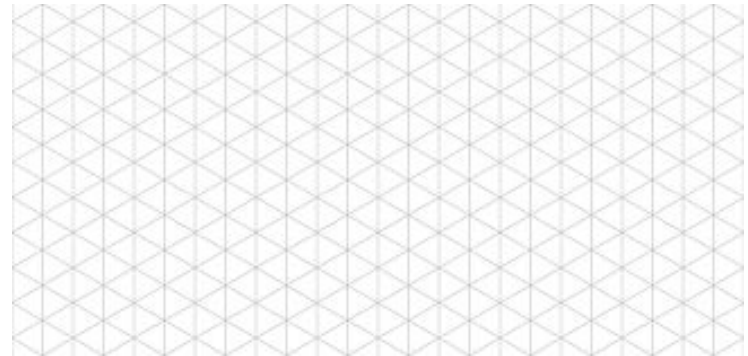
Lemmas we will use

Theorem 1.2. (Erdős et. al.) **Will Not Prove**

Lemma 2.1. If E^2 is 2-colored without a monochromatic L_3 , any unit eq. triangle colored R-R-B has B centroid **Just Proved**

Lemma 2.2. If E^2 is 2-colored without a monochromatic L_3 , a $\frac{1}{\sqrt{3}}$ scaled

Hexagonal grid has one valid coloring



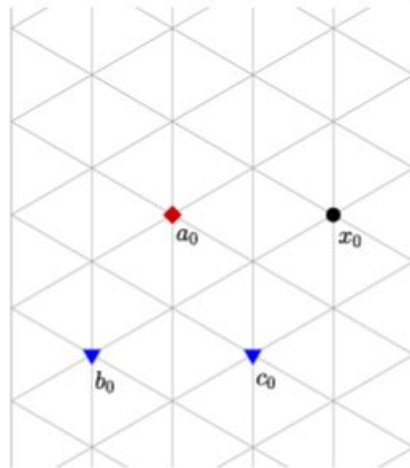
Proof for Lemma 2.2:

If E_2 is 2-colored without a monochromatic L_3 , a $\frac{1}{\sqrt{3}}$ scaled

Hexagonal grid has one valid coloring

W.L.O.G. Suppose a unit triangle on points $\{a_0, b_0, c_0\}$ with R,B,B coloring, respectively

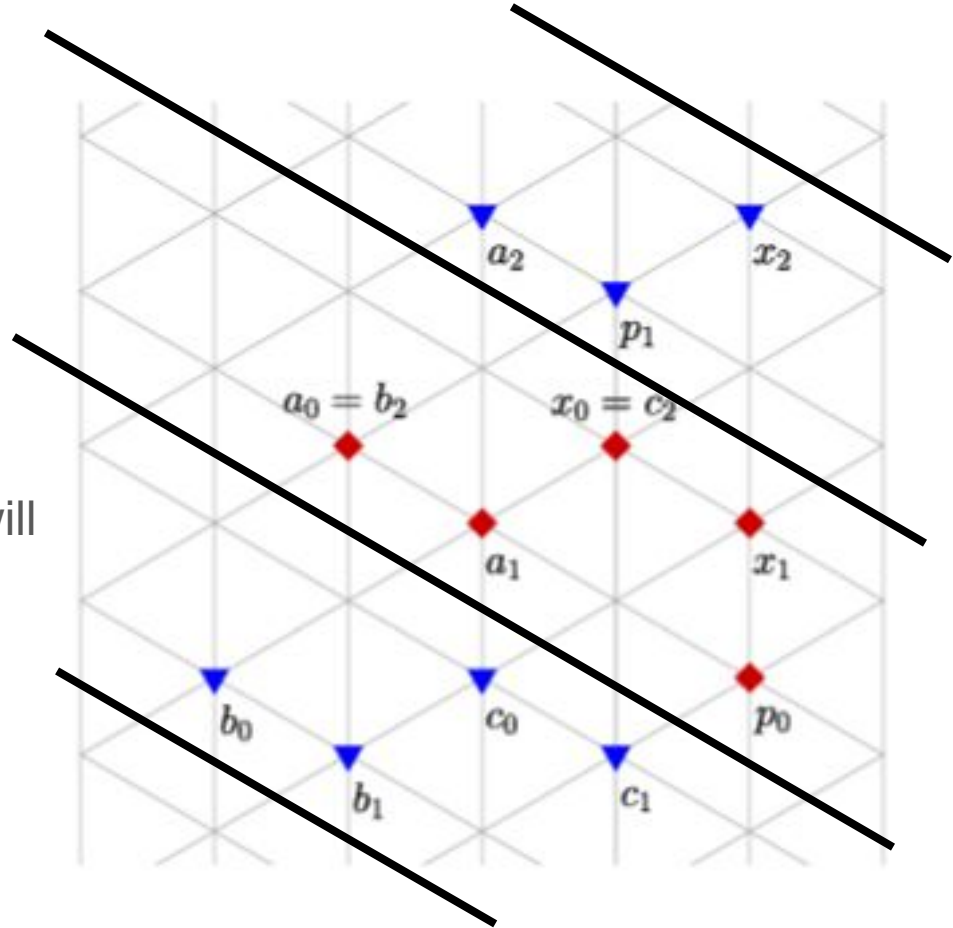
Now define x_0 as:



Casework again on x_0

If x_0 is R:

By continuing the steps, the grid will
result in repetition of:
2 rows of R and 2 rows of B

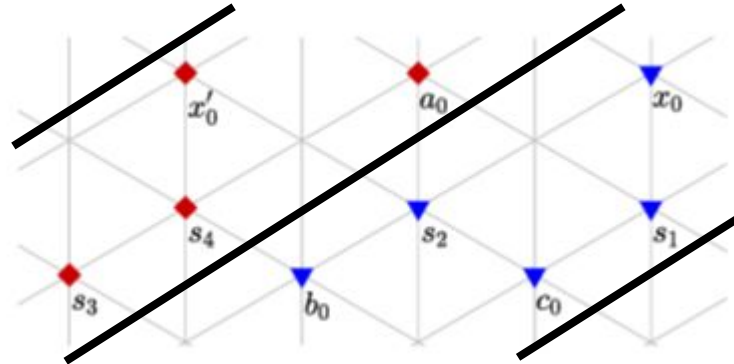


Casework again on x_0

If x_0 is B:

By repetition, it will result the same
as when x_0 is R

End of proof for Lemma 2.2.



Using Lemma 2.2 repeatedly results...

A unique grid!

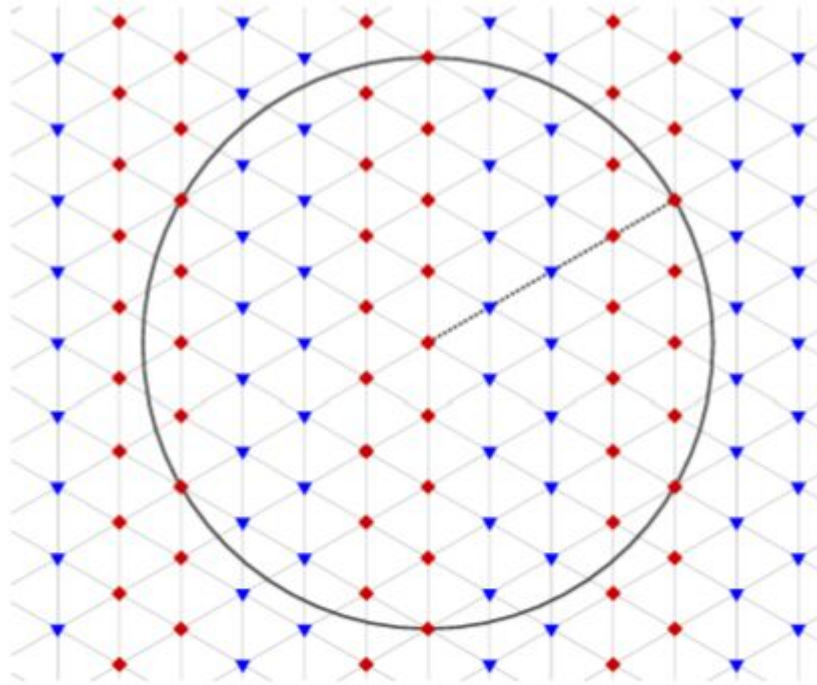


FIGURE 7. A circle with radius $\frac{4}{\sqrt{3}}$ in the colored grid

Proof of Theorem 1.1.

From the unique grid,

Now choose a R point:

All 6 points $\frac{4}{\sqrt{3}}$ units away from
the point are also red

Rotate the whole grid to make a
red circle of radius $\frac{4}{\sqrt{3}}$

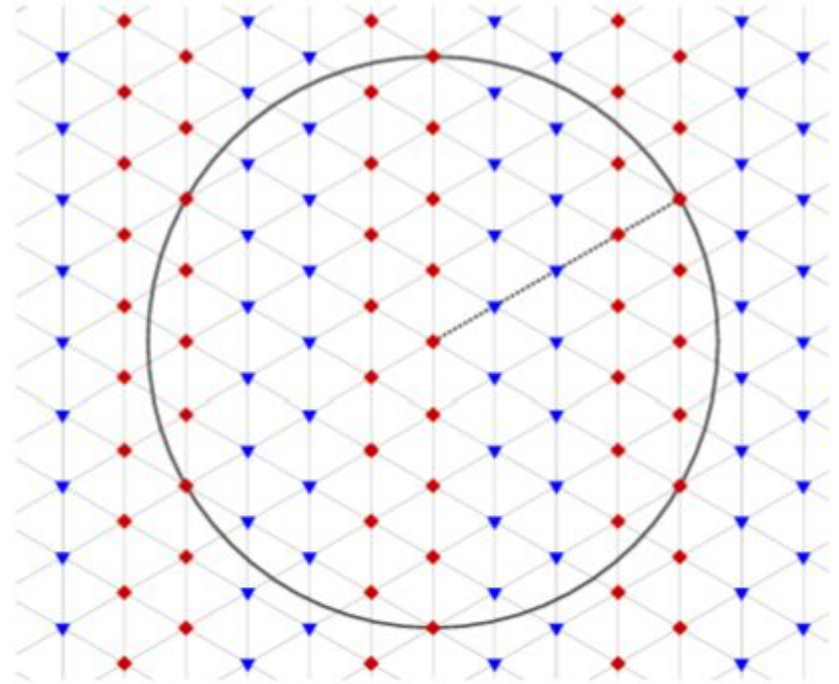


FIGURE 7. A circle with radius $\frac{4}{\sqrt{3}}$ in the colored grid

Now take a B point close enough
and make a congruent circle with
B color, which would intersect
with R circle

The intersection are both B and R

Contradiction!!

End of the proof for the Theorem 1.1

