The Muffin Problem - Data Analysis of Exceptions

By Dan Smolyak

1 Introduction

Throughout the previous sections of this paper, we have discussed methods of finding $f(m, s)$ and specifically, finding what $f(m, s)$ is when the Floor-Ceiling Theorem, and the patterns determined by it, aren't matched by the actual $f(m, s)$. We now analyze these exceptions to the Floor-Ceiling theorem, investigating the following aspects:

- How often do exceptions occur?
- For which m do these exceptions occur? (Given a single s)
- What is the maximum m for which exceptions occur? (Given a single s)

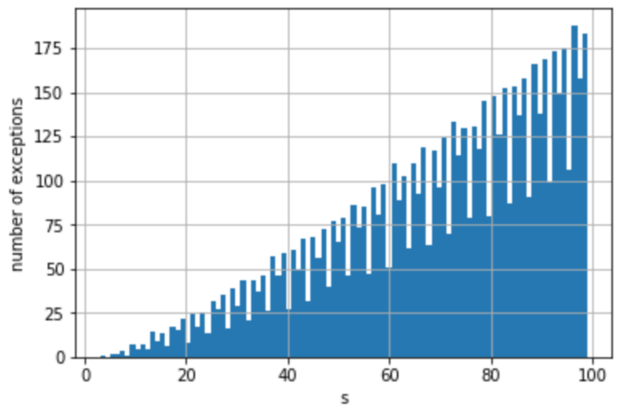
2 The Muffin Data

As a brief explanation, we collected data by first running all of the various exception-detecting programs on $f(m, s)$, for s ranging from 0 to 60, where for each s, we tested each m from $s + 1$ to s^2. We chose s^2 as an upper bound relatively arbitrarily, but you'll see later on that it was a good starting bound. Once we found the actual approximate bounds on the maximum m with an exception for a given s, we used these bounds to find more data for s from 61 to 100, thus bootstrapping to find new results.

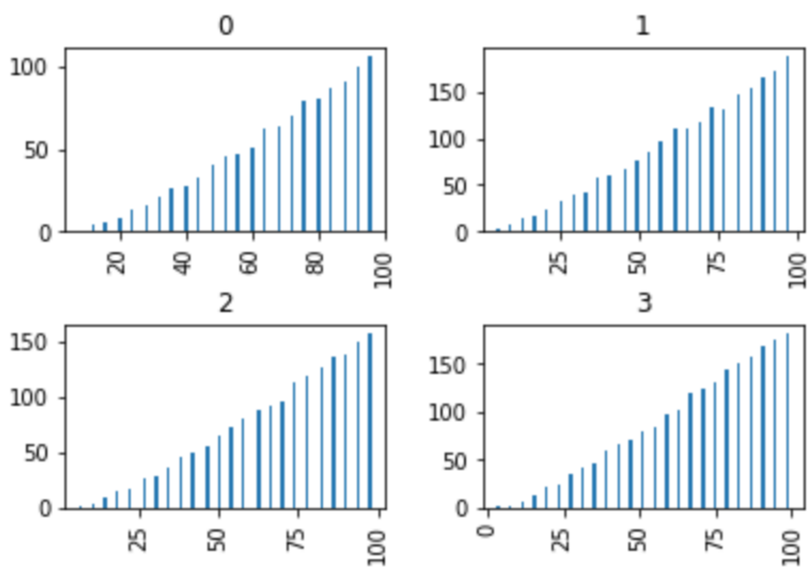
3 Frequency of Exceptions

As s increases, the number of exceptions tends to increase linearly. Within each s mod 4, the increase is linear, but there are spikes up and down between different s mod 4.

Below is the number of exceptions per s:



From above, we see mod patterns where certain s have consistently higher/lower number of exceptions. The graph below shows the number of exceptions at each s mod 4. Make sure to look at the y-axis for each graph.



We can now see the directly linear trend in each s mod 4. Notice 1 and 3 are the same, while 2 is slightly smaller, and 0 is much smaller. We will see the same trend in the next section as well.

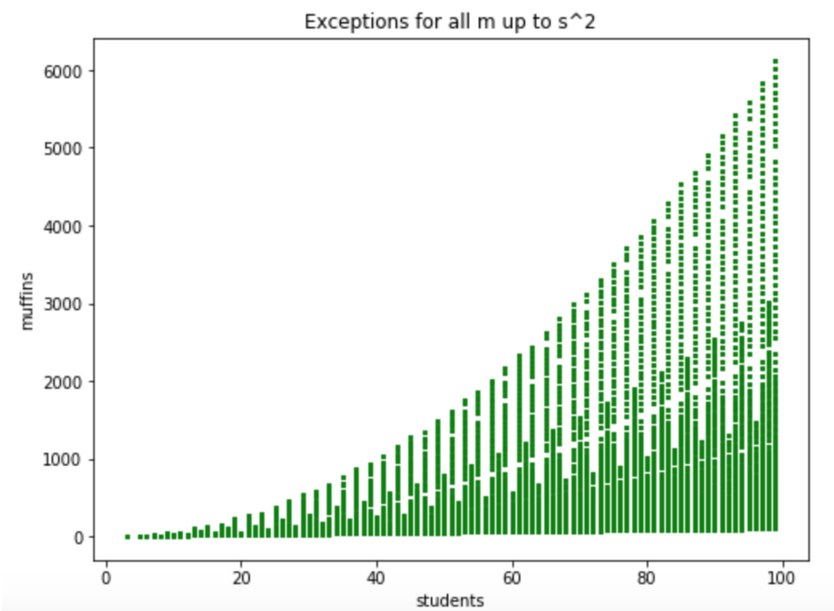
Below are the actual equations for the linear trend of each s mod 4 (the R-squared for each of those was $\geq .98$):

s	$freq$
$s \bmod 2 \equiv 1$	$\mathbf{2.00}s - 17.08$
$s \bmod 4 \equiv 2$	$\mathbf{1.75}s - 18.75$
$s \bmod 4 \equiv 0$	$\mathbf{1.21}s - 16.10$

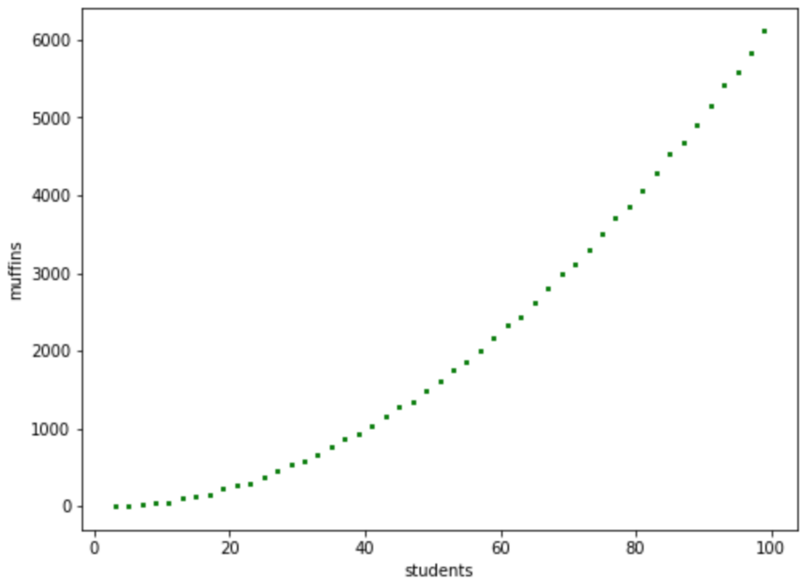
4 Maximum Exceptions

The maximum m with an exception increases quadratic with s. Once again, there are differences between each s mod 4.

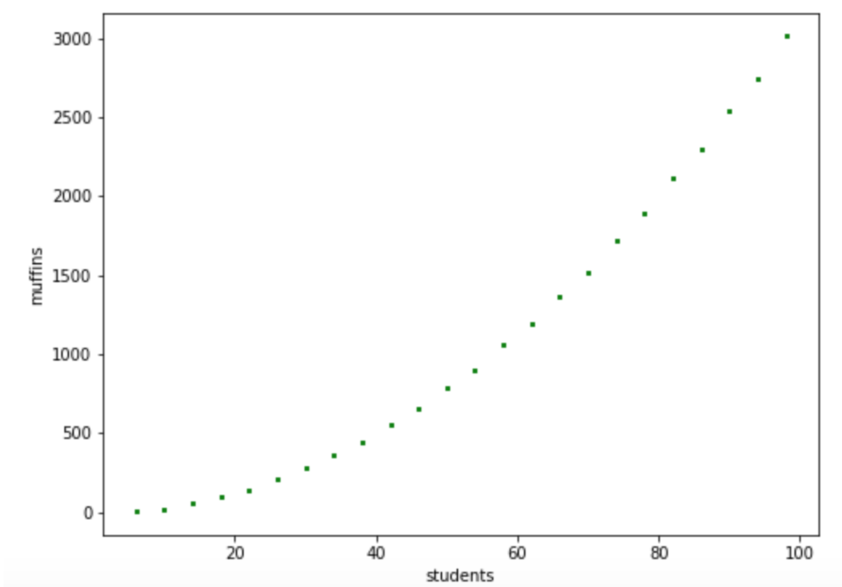
Below we plot all of the exceptions for each s:



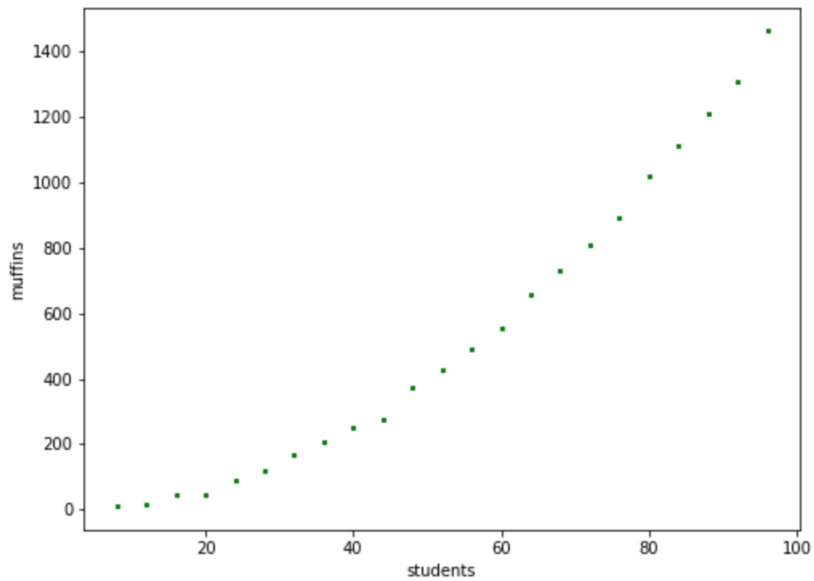
Once again, a mod 4 pattern seemed to appear, with the odd s having exceptions appearing at consistently higher m. We now plot the odd s:



Now the s mod $4 \equiv 2$:



Now the s mod $4 \equiv 0$:



We then ran polynomial regression on each of these cases and found the below constants (The R-squared for each of these was ≥ 0.999):

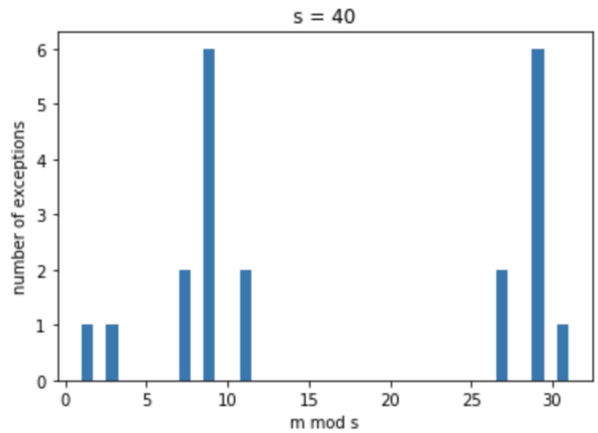
s	$freq$
$s \bmod 2 \equiv 1$	$\mathbf{.63}s^2 + 0.00s - 8.31$
$s \bmod 4 \equiv 2$	$\mathbf{.31}s^2 + 0.11s - 5.33$
$s \bmod 4 \equiv 0$	$\mathbf{.16}s^2 - 0.03s - 3.16$

From above, it is clear that the s^2 constant differs drastically for each mod, while the s and intercept terms are more arbitrary. While we don't know why, there does seem to be a halving in the s^2 term for each mod.

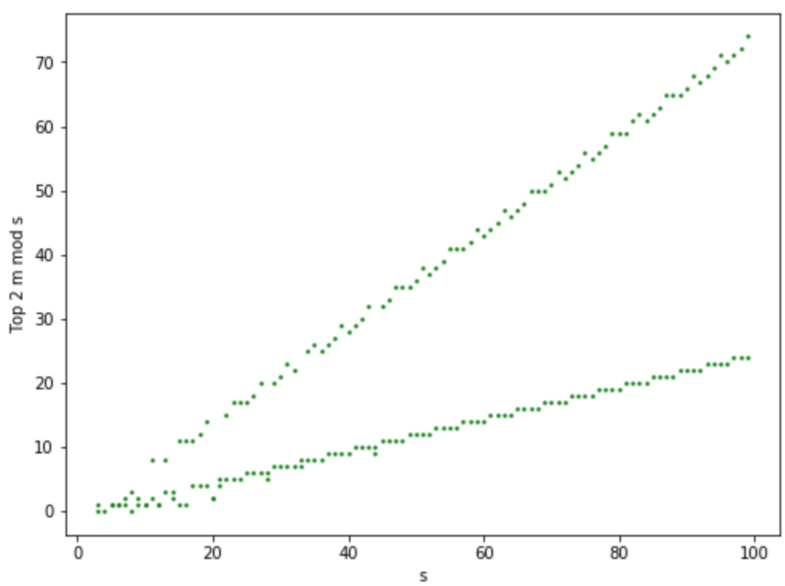
5 Common m's for Exceptions

The two most common m mod s for exceptions to occur, for a given s, increase linearly. Alan Frank, the original creator of the Muffin problem, had conjectured that there was a relation between s and the m mod s where most exceptions occurred. This indeed was the case.

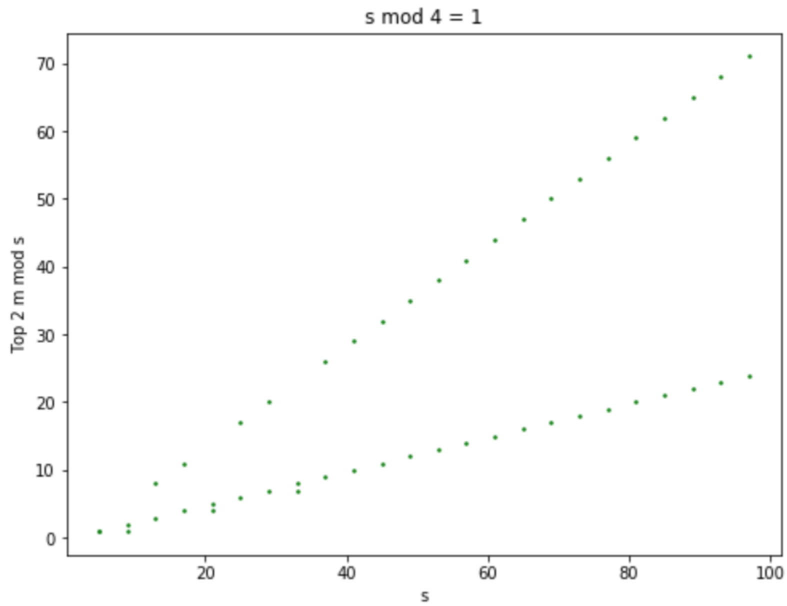
Below we show the m mod s at which exceptions occur for $s = 40$:



As you can see there are two major spikes at certain m mod s. Specifically, $f(m, 40)$ when m mod $40 \equiv 9$ $(m = 49, 89, \ldots)$ and when m mod $40 \equiv 29$ $(m = 69, 109, \ldots)$ are more likely to be exceptions than other m mod s. Below is plotted the two m mod s for each s, that had the largest number of exceptions:



These two values clearly increase linearly with s. To get the exact trend, we once again separated s values out by their value of s mod 4. Below is the plot for s mod $4 \equiv 1$ (the others look very similar):



Once we separated these values, trimmed any values below a certain threshold ($s < 20$ for $s \bmod 4 \equiv 0$ but lower threshold for others), and took only the maximum value, if two appeared for the smaller trend, we then ran linear regression.

These are the expressions for the lower trend (R-squared = 1):

$s \bmod 4 \equiv \dots$	$m_{max} = \dots$
0	$\mathbf{.25}s - 1.00$
1	$\mathbf{.25}s - 0.25$
2	$\mathbf{.25}s - 0.50$
3	$\mathbf{.25}s - 0.75$

These are the expressions for the upper trend (R-squared ≥ 0.999):

$s \bmod 4 \equiv \dots$	$m_{max} = \dots$
0	$\mathbf{.75}s - 1.55$
1	$\mathbf{.75}s - 1.75$
2	$\mathbf{.75}s - 1.40$
3	$\mathbf{.75}s - 0.25$

Clearly, these are the overall trends:

$$small \rightarrow m_{max} = \mathbf{.25}s$$

$$large \rightarrow m_{max} = \mathbf{.75}s$$

Thus, when m mod s is approximately $.25s$ or $.75s$ there are going to be greater numbers of exceptions for those $f(m, s)$, such as with m mod $s \equiv 9$ or m mod $s \equiv 29$ when $s = 40$.