The Muffin Problem

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How it Began

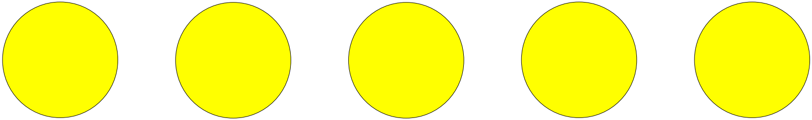
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

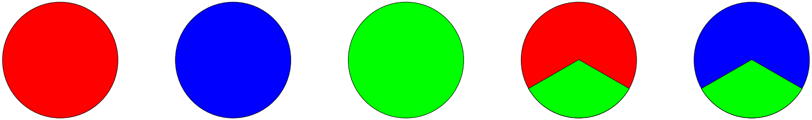
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ *where nobody gets a tiny sliver?*



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

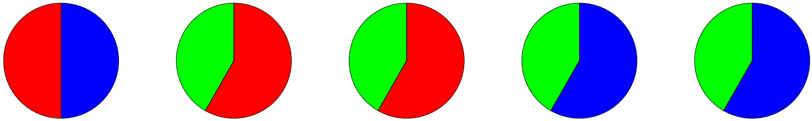
Work on it with your neighbor

Five Muffins, Three People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor

5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \le \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

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(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

What Happened Next?

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Yada Yada Yada- in 2020:

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General Problem

$f(m,s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have two proofs that shown $f(m,s)$ exists, is rational, and is computable.
One use Linear Programming.
One use Integer Programming.

Amazing Results!/Amazing Theorems!

1. $f(43,33) = \frac{91}{264}$.
2. $f(52,11) = \frac{83}{176}$.
3. $f(35,13) = \frac{64}{143}$.

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant !

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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$.
Can we get $f(3,5) > \frac{1}{5}$?
Work on it with your neighbor

$f(3,5) \geq \frac{1}{4}$

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1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

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4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

Work on it with your neighbor

Three Muffins, Five People–Can't Do Better Than $\frac{1}{4}$

NO WE CAN'T!

There is a procedure for 3 muffins,5 students where each student gets $\frac{3}{5}$ muffins, smallest piece N. We want $N \le \frac{1}{4}$.

Case 0: Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\le \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.

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(**Henceforth:** All people get ≥ 2 pieces.)

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Case 1: Alice gets ≥ 3 pieces. Then $N \le \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$.
(**Henceforth:** Everyone gets 2 pieces.)

Case 2: Everyone gets 2 pieces. 10 pieces, 3 muffins:
Some muffin gets ≥ 4 pieces. So some piece is $\le \frac{1}{4}$.

$f(3,5)$ and $f(5,3)$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5)$ and $f(5,3)$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

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1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
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1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by $3/5$.

$f(3,5)$ and $f(5,3)$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
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1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
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$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by 3/5.
Duality Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

Conventions

We know and use the following:

1. By duality can assume $m > s$
2. If s divides m then $f(m, s) = 1$ so assume s does not divide m.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s
4. If assuming $f(m, s) > \alpha > \frac{1}{3}$, assume all muffin in ≤ 2 pcs.
5. $f(m, s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

We do not know this but still use: $f(m, s)$ only depends on $\frac{m}{s}$.
All of our techniques that hold for (m, s) hold for (Am, As).
For particular numbers, we only look at m, s rel prime.

What if $m < s$?

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Duality Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

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Duality Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.
Hence we will just look at $m > s$.

Floor-Ceiling Thm Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

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Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

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Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\le \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{\mathbf{m}}{\mathbf{s}\lceil \mathbf{2m/s}\rceil}$.

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Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \mathbf{\frac{m}{s\lceil 2m/s\rceil}}$.

Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

FC Gives Upper Bound

Give m, s:

$$\mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s \rceil}, 1 - \frac{m}{s\lfloor 2m/s \rfloor}\right\}\right\}$$

Gives an upper bound. So we know

$$(\forall m, s)[f(m,s) \le \mathrm{FC}(m,s)].$$

Is the following true?

$$(\forall m, s)[f(m,s) = \mathrm{FC}(m,s)]$$

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Is the following true?

$$(\forall m,s)[f(m,s) = \mathrm{FC}(m,s)]$$

No: If so my book would be about 20 pages.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1$.

$f(3k+1,3) = \frac{3k-1}{6k}$, $k \geq 1$.

$f(3k+2,3) = \frac{3k+2}{6k+6}$.

Note: A Mod 3 Pattern.
Theorem: For all $m \geq 3$, $f(m,3) = \mathrm{FC}(m,3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \mathrm{FC}(m, 4)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \mathrm{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k+3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.
Theorem: For all $m \geq 5$ **except m=11**, $f(m, 5) = \mathrm{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11,5) \geq \frac{13}{30}$.
2. $f(11,5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote

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Vote WE SHOW $f(11,5) = \frac{13}{30}$. **Exciting** new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the **buddy of x**.

Note that the **buddy** of x is of size

$$1 - x.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

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(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

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$$\ge \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

That pieces **buddy** is of size:

$$\le 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

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That pieces **buddy** is of size:

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

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$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

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$s_4 = 3$: There are 3 students who have 4 shares.
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We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets w, x, y, z and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

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The **buddy** of x is of size

$$\le 1 - x = 1 - \frac{17}{30} = \frac{13}{30}$$

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GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

HALF Method

The above reasoning can be used to *verify* that $f(11,5) \le \frac{13}{30}$ but could not *generate* the upper bound $\frac{13}{30}$.

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For $f(24,11)$ it fails!

$f(24,11) \le \frac{19}{44}$

Assume $(24,11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\le \frac{19}{44}$.

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Case 1: A student gets ≥ 6 shares. Some piece $\le \frac{24}{11\times 6} < \frac{19}{44}$.

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Case 2: A student gets ≤ 3 shares. Some piece $\ge \frac{24}{11\times 3} = \frac{8}{11}$.
Buddy of that piece $\le 1 - \frac{8}{11} \le \frac{3}{11} < \frac{19}{44}$.

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Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.
5-students: a student who gets 5 shares. s_5 is the number of them.

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$$
\begin{aligned}
4s_4 + 5s_5 &= 48 \\
s_4 + s_5 &= 11
\end{aligned}
$$

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$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: $\exists$ a share $\geq \frac{25}{44}$. Its **buddy** is

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Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44}\right)$$

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Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

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Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.
Proof: Assume that Alice 5 pieces A, B, C, D, E and $E \geq \frac{20}{44}$.
Since $A + B + C + D + E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A + B + C + D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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Assume A is the smallest of A, B, C, D.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

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Henceforth we assume all 5-shares are in

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Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 4 pieces A, B, C, D and $D \leq \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$

$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

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$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume A is the largest of A, B, C.

$$A \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The **buddy** of A is of size

$$\le 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.4: Some 4-shares $\le \frac{21}{44}$

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Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.
Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

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More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(}\quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[}\quad \text{0 shs} \quad \underset{\frac{21}{44}}{](}\quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

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Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

More Refined Picture of What is Going On

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If there was such a share then its **buddy** is in $[\frac{20}{44}, \frac{21}{44}]$.

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The following picture captures what we know so far.

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

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Claim 2: Every 4-student has at least 3 L4 shares.

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Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

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Contradiction: Each 4-student gets ≥ 3 L4 shares. There are $s_4 = 7$ 4-students. Hence there are ≥ 21 L4-shares. But there are only 20.

INT Technique

The above reasoning can be used to *verify* that $f(24, 11) \leq \frac{19}{44}$ but could not *generate* the upper bound $\frac{19}{44}$.

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For $f(31, 19)$ it fails!

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We show $f(31,19) \le \frac{54}{133}$.
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Assume $(31,19)$-procedure with smallest piece $> \frac{54}{133}$.
By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

(	20 4-shs	)[	0	](	S3 shs	)[	0	](	20 L3-shs	)
$\frac{54}{133}$		$\frac{55}{133}$		$\frac{59}{133}$		$\frac{74}{133}$		$\frac{78}{133}$		$\frac{79}{133}$

$f(31,19) \le \frac{54}{133}$

We show $f(31,19) \le \frac{54}{133}$.
Assume (31, 19)-procedure with smallest piece $> \frac{54}{133}$.
By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

(	20 4-shs	)[	0	](	S3 shs	)[	0	](	20 L3-shs	)
$\frac{54}{133}$		$\frac{55}{133}$		$\frac{59}{133}$		$\frac{74}{133}$		$\frac{78}{133}$		$\frac{79}{133}$

We just look at the 3-shares:

(	S3 shs	)[	0	](	20 L3-shs	)
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$\frac{59}{133}$		$\frac{74}{133}$		$\frac{78}{133}$		$\frac{79}{133}$

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ $(|J_3| = 20)$

Note: Split the shares of size 66.5 between J_1 and J_2.

Notation: An $e(1,1,3)$ students is a student who has a J_1-share, a J_1-share, and a J_3-share.

Generalize to $e(i,j,k)$ easily.

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1) Only students allowed: $e(1,2,3)$, $e(1,3,3)$, $e(2,2,2)$, $e(2,2,3)$. All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

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2) No shares in $[\frac{61}{133}, \frac{64}{133}]$. Look at J_1-shares:
An $e(1, 2, 3)$-student has J_1-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
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3) No shares in $[\frac{69}{133}, \frac{72}{133}]$: $x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}]$.

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4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ $(|J_1| = |J_4|)$
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The following are the only students who are allowed.
$e(1,5,5)$.
$e(2,4,5)$,
$e(3,4,5)$.
$e(4,4,4)$.

$f(31,19) \leq \frac{54}{133}$

$e(1,5,5)$. Let the number of such students be x
$e(2,4,5)$. Let the number of such students be y_1
$e(3,4,5)$. Let the number of such students be y_2.
$e(4,4,4)$. Let the number of such students be z.

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Hence $y_1 = y_2$. We call them both y.

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$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.

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Cannot quite modify the method, but we can use this method and a method we have to find procedure and to a binary search to zero-in on answer. We call this $\mathrm{GAP}(m,s)$. So we have

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For $f(67, 21)$ it fails!

The Train Method

We developed the Train Method which showed settled $f(67, 21)$ and 13 other problems we could not do.

Upshot

Let

$$A = \{(m, s) \mid 2 \le s \le 100 \text{ and } s < m \le 110 \text{ and } m, s \text{ rel prime}\}$$

There are 3520 pairs (m, s) in A. We solved **all** of them!

- For 2301 of them $f(m, s) = \mathrm{FC}(m, s)$. That is $\sim 65.37\%$.
- For 329 of them $f(m, s) = \mathrm{HALF}(m, s)$. That is $\sim 9.35\%$.
- For 186 of them $f(m, s) = \mathrm{INT}(m, s)$. That is $\sim 5.28\%$.
- For 111 of them $f(m, s) = \mathrm{MID}(m, s)$. That is $\sim 3.15\%$.
- For 240 of them $f(m, s) = \mathrm{EBM}(m, s)$. That is $\sim 6.28\%$.
- For 89 of them $f(m, s) = \mathrm{HBM}(m, s)$. That is $\sim 2.53\%$.
- For 250 of them $f(m, s) = \mathrm{GAP}(m, s)$. That is $\sim 7.10\%$.
- For 13 of them $f(m, s) = \mathrm{TRAIN}(m, s)$. That is $\sim 0.40\%$

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2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}$, $\{\frac{5}{12}, \frac{7}{12}\}$. Vectors
$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0,2,0)$, m_1 muffins of this type.
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$m_1(0,2,0) + m_2(1,0,1) = s_1(0,1,2) + s_2(4,0,0)$

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Natural Number Solution: $m_1 = 1$, $m_2 = 4$, $s_1 = 2$, $s_2 = 1$

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$m_1\vec{v_1} + \cdots + m_x\vec{v_x} = s_1\vec{u_1} + \cdots + s_y\vec{u_y}$
$m_1 + \cdots + m_x = m$
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5) **Look for Nat Numb sol.** If find can translate into procedure.

So Where Are We Now?

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No. Did not work on

- $f(205, 178)$
- $f(226, 135)$
- $f(233, 141)$

The Scott Huddleston Technique

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Richard Chatwin independently came up with the same algorithm and proved it worked, but never coded it up. He tells me its poly time and I believe this can be proved, though its not in his paper. His paper is here: `https://arxiv.org/abs/1907.08726`

Interesting VS Important

A Math problem can be

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- Important or NOT Important.

These are **not** well defined terms but

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Important Either useful OR about deep mathematical objects (hard to define that).

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4. Not Interesting and Not Important: So far $R(5)$ has not lead to any math of interest and is also not important to find. Same for most Ramsey-type Numbers.

Lessons Learned

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You never know where the next big project will come from!