The Muffin Problem

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How it Began

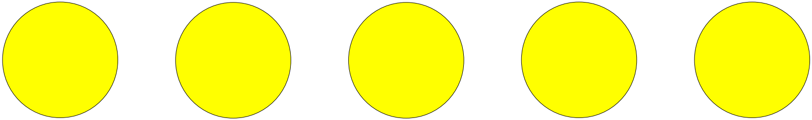
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by **Alan Frank**:

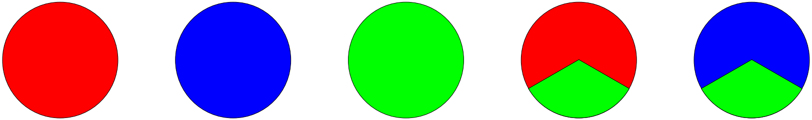
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



5 Muffins, 3 Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

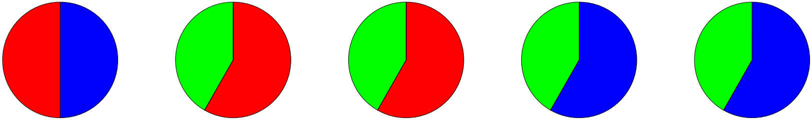
The smallest piece in the above solution is $\frac{1}{3}$.
Is there a procedure with a larger smallest piece?
Work on it with your neighbor

5 Muffins, 3 People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Carol	GREEN	$\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor

5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into $\geq \mathbf{2}$ pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins cut into 2 pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

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- Find a technique that solves many problems (e.g., Floor-Ceiling).

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- Find a technique that solves many problems (e.g., Floor-Ceiling).
- Come across a problem where the techniques do not work.
- Find a new technique **which was interesting**.
- Lather, Rinse, Repeat.

General Problem

$\boldsymbol{f(m,s)}$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have shown $f(m,s)$ exists, is rational, and is computable using a **Mixed Int Program**.

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We have shown $f(m,s)$ exists, is rational, and is computable using a **Mixed Int Program**.
This was a case of a Theorem in **Applied Math** being used to prove a Theorem in **Pure Math**.

Amazing Results!/Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

Conventions

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We know and use the following:

1. By Duality Theorem can assume $m > s$
2. By REASONS we can assume m, s are relatively prime.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s

7 Muffins, 3 Students

Work on $f(7,3)$ in groups.
7 Muffins, 3 Students.
Get upper and lower bounds that match!

7 Muffins, 3 Students: How to think about it

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Now what?

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6. That piece came from a muffin. Other piece is $\leq 1 - \frac{7}{12} = \frac{5}{12}$.

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7. Great! We know $f(7,3) \leq \frac{5}{12}$.
8. Can we show a protocol that gives $f(7,3) \geq \frac{5}{12}$?

7 Muffins, 3 Students: How to think about protocol

Want $f(7,3) \geq \frac{5}{12}$.

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Want $f(7,3) \ge \frac{5}{12}$.
Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

7 Muffins, 3 Students: How to think about protocol

Want $f(7,3) \geq \frac{5}{12}$.
Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.
Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.
Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.
Need to know what combos of $5, 6, 7$ add to 28.
$7 + 7 + 7 + 7 = 28$
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2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.
4. Give 2 students 2 pieces of size $\frac{5}{12}$ and 3 pieces of size $\frac{6}{12}$.

8 Muffins, 3 Students

Work on $f(8,3)$ in groups.
8 Muffins, 3 Students.
Get upper and lower bounds that match!

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1. If a muffin is uncut, can cut it in two.
2. If a muffin is cut in ≥ 3 pieces then some piece $\leq \frac{1}{3}$. Unlikely that thats a good idea.
3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.

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4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.

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5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.

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5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.
6. Great! We know $f(8,3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.

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3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.
4. 3 students, so some student gets $\ge \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\le \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.
5. 3 students, so some student gets $\le \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\ge \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\le 1 - \frac{8}{15} = \frac{7}{15}$.
6. Great! We know $f(8,3) \le \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.
7. Can we show a protocol that gives $f(8,3) \ge \frac{4}{9}$?

8 Muffins, 3 Students: How to think about protocol

Want $f(8,3) \geq \frac{4}{9}$.

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Want $f(8,3) \geq \frac{4}{9}$.
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8 Muffins, 3 Students: How to think about protocol

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Will be cutting some muffins $(\frac{4}{9}, \frac{5}{9})$.
$\frac{1}{2}$ was helpful last time so lets also include $\frac{4.5}{9}$.
Need to know what combos of $\frac{4}{9}, \frac{4.5}{9}, \frac{5}{9}$ add to $\frac{8}{3} = \frac{24}{9}$.

8 Muffins, 3 Students: How to think about protocol

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Need to know what combos of $4, 4.5, 5$ add to 24

$4 + 4 + 4 + 4 + 4 + 4 = 24$

$4.5 + 4.5 + 5 + 5 + 5 = 24$

8 Muffins, 3 Students: How to think about protocol

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Need to know what combos of $4, 4.5, 5$ add to 24
$4+4+4+4+4+4=24$
$4.5+4.5+5+5+5=24$

1. Cut 6 muffins $(\frac{4}{9}, \frac{5}{9})$.
2. Cut 2 muffins $(\frac{4.5}{9}, \frac{4.5}{9})$.
3. Give 1 student six $\frac{4}{9}$ pieces.
4. Give 2 students two $\frac{4.5}{9}$ pieces and four $\frac{5}{9}$ pieces.

FC Thm Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

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Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

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Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{m}{s\lceil 2m/s\rceil}$.

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Someone gets $\le \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

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$f(1,3) = \frac{1}{3}$

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$f(3k+1,3) = \frac{3k-1}{6k}$, $k \geq 1.$

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$f(3k+2,3)=\frac{3k+2}{6k+6}$.

Note: A Mod 3 Pattern.
Theorem: For all $m\ge 3$, $f(m,3)=\mathrm{FC}(m,3)$.

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$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

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$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

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$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \mathrm{FC}(m, 4)$.

FOUR Students

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FC Theorem for optimality.

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$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \mathrm{FC}(m, 4)$.
FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = \mathrm{FC}(m, s)$.

FIVE Students

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FC Theorem for optimality.

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FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

FIVE Students

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FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

FIVE Students

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FC Theorem for optimality.

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For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \mathrm{FC}(7, 5) = \frac{1}{3}$

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For $k \ge 1$, $f(5k+3,5) = \frac{5k+3}{10k+10}$

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For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.

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FC Theorem for optimality.

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For $k \geq 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.
Theorem: For all $m \geq 5$ **except m=11**, $f(m, 5) = \mathrm{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

$$f(11,5) \le \max\left\{\frac{1}{3}, \min\left\{\frac{11}{5\left\lceil 22/5\right\rceil}, 1 - \frac{11}{5\left\lfloor 22/5\right\rfloor}\right\}\right\} = \frac{11}{25}.$$

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.
We found a protocol with smallest piece $\frac{13}{30} = 0.4333\ldots$.

1. Divide 1 muffin $(\frac{15}{30}, \frac{15}{30})$.
2. Divide 2 muffins $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 8 muffins $(\frac{13}{30}, \frac{17}{30})$.
4. Give 2 students $[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30}]$
5. Give 1 students $[\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]$
6. Give 2 students $[\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]$

So Now What?

We have:

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff}= 0.006666\ldots$$

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Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote

So Now What?

We have:

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff}= 0.006666\ldots$$

Options:

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Vote WE SHOW: $\mathbf{f(11,5) = \frac{13}{30}}$. **Exciting** new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the *buddy of* x.

Note that the buddy of x is of size

$$1 - x.$$

$f(11,5)=\frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

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There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5)=\frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

$f(11,5)=\frac{13}{30}$, Easy Case Based on Students

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One of the pieces is

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Look at the muffin it came from to find a piece that is

$$\le 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

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Look at the muffin it came from to find a piece that is

$$\le 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

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$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

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$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$f(11,5)=\frac{13}{30}$, Fun Cases

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$$x+y+z \ge \frac{11}{5}-\frac{1}{2}=\frac{17}{10}$$

$$z \ge \frac{17}{10}\times\frac{1}{3}=\frac{17}{30}$$

$f(11,5) = \frac{13}{30}$, Fun Cases

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Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
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$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z.

$$B(z) \le 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
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$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

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Look at **buddy** of z.

$$B(z) \le 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

INT Method

Proof that $f(11,5) \le \frac{13}{30}$ was an example of the HALF method.

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FC or HALF worked on everything with $s = 3, 4, 5, \ldots, 23$.

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Then we found a case where neither FC nor HALF worked.

INT Method

Proof that $f(11,5) \le \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $s = 3, 4, 5, \ldots, 23$.

Then we found a case where neither FC nor HALF worked.

We found a new method: INT.

More Sophisticated INT: $f(24, 11) \le \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
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Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

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Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.
5-students: a student who gets 5 shares. s_5 is the number of them.

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$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

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Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44}\right)$$

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Case 3.3: Some 5-shares $\ge \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\ge \frac{20}{44}$ then some share $\le \frac{19}{44}$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

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Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

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$$v+w+x+y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

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Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

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4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

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4-share: a share that a 4-student who gets.

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Since $w + x + y + z = \frac{24}{11}$ and $w \le \frac{21}{44}$

$$x + y + z \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

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Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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More Refined Picture of What is Going On

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Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

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The following picture captures what we know so far.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

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S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

Diagram

$$\underset{\frac{19}{44}}{(}\quad 20\ \text{5-shs}\quad \underset{\frac{20}{44}}{)[}\quad 0\quad \underset{\frac{21}{44}}{](}\quad 8\ \text{S4-shs}\quad \underset{\frac{23}{44}}{)[}\quad 0\quad \underset{\frac{24}{44}}{](}\quad 20\ \text{L4-shs}\quad \underset{\frac{25}{44}}{)}$$

Diagram

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
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Claim 2: Every 4-student has at least 3 L4 shares.

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If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

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There are $s_4 = 7$ 4-students.
Hence there are ≥ 21 L4-shares.

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Hence there are ≥ 21 L4-shares. But there are only 20.

GAPS Method

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We found a new method: GAP.

Example of GAPS Technique: $f(31, 19) \le \frac{54}{133}$

We show $f(31, 19) \le \frac{54}{133}$.
Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$.

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By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

$$\underset{\frac{54}{133}}{(} \quad \text{20 4-shs} \quad \underset{\frac{55}{133}}{)[} \quad 0 \quad \underset{\frac{59}{133}}{](} \quad \text{22 S3 shs} \quad \underset{\frac{74}{133}}{)[} \quad 0 \quad \underset{\frac{78}{133}}{](} \quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

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1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
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Notation: An $e(1,1,3)$ students is a student who has a J_1-share, a J_1-share, and a J_3-share.

Generalize to $e(i,j,k)$ easily.

GAPS Technique: $f(31,19) \le \frac{54}{133}$

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Notation: An $e(1,1,3)$ students is a student who has a J_1-share, a J_1-share, and a J_3-share.

Generalize to $e(i,j,k)$ easily.

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GAPS Technique: $f(31,19) \le \frac{54}{133}$

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1) Only students allowed: $e(1,2,3)$, $e(1,3,3)$, $e(2,2,2)$, $e(2,2,3)$.
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An $e(1,2,3)$-student has J_1-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
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4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ $(|J_1| = |J_4|)$
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The following are the only students who are allowed.
$e(1,5,5)$.
$e(2,4,5)$,
$e(3,4,5)$.
$e(4,4,4)$.

GAPS Technique: $f(31,19) \le \frac{54}{133}$

$e(1,5,5)$. Let the number of such students be x
$e(2,4,5)$. Let the number of such students be y_1
$e(3,4,5)$. Let the number of such students be y_2.
$e(4,4,4)$. Let the number of such students be z.

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$e(3,4,5)$. Let the number of such students be y_2.
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1) $|J_2| = |J_3|$,
only students using J_2 are $e(2,4,5)$ – they use one share each,
only students using J_3 are $e(3,4,5)$ – they use one share each.
Hence $y_1 = y_2$. We call them both y.

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2) Since $|J_1| = |J_4|$, $x = 2y + 3z$.

3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.

MATRIX Technique: $f(5,3) \geq \frac{5}{12}$

Want proc for $f(5,3) \geq \frac{5}{12}$.

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2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}$, $\{\frac{5}{12}, \frac{7}{12}\}$. Vectors
$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0,2,0)$, m_1 muffins of this type.
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3) **Student**=pieces add to $\frac{5}{3}$
$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0,1,2)$, s_1 students of this type.
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4) **Set up equations:**
$m_1(0,2,0) + m_2(1,0,1) = s_1(0,1,2) + s_2(4,0,0)$
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Natural Number Solution: $m_1 = 1$, $m_2 = 4$, $s_1 = 2$, $s_2 = 1$

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s_j students of type $\vec{u_j}$

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$m_1\vec{v_1} + \cdots + m_x\vec{v_x} = s_1\vec{u_1} + \cdots + s_y\vec{u_y}$
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5) **Look for Nat Numb sol.** If find can translate into procedure.

Later Results by Other People

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5. One corollary of the work: $f(m, s)$ only depends on m/s.

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How it worked

We kept increasing s.

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This happened 7 times leading to techniques now called:
Floor Ceiling, Half, Int,
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How it worked

We kept increasing s.

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Also a chapter that sketched out Scott H's method.

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