Quantum Bits, Entanglement, and the CHSH Game

Exposition by
William Gasarch and Evan Golub

January 7, 2025

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5. We give a strategy for the CHSH game where (1) the 2 players have qubits that are entangled, and (2) **the prob of winning is larger than 0.75.**

Quantum Bits I: Measure Once

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On next slide we show that M_θ is unitary.

Proof that M_θ is Unitary

Let $v = (\alpha, \beta)$ be a vector. We show $N(M_\theta(v)) = N(v)$.

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos(\theta)\alpha - \sin(\theta)\beta \\ \sin(\theta)\alpha + \cos(\theta)\beta \end{pmatrix}$$

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&= \alpha^2 + \beta^2
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Def A **qubit** is something in physics that has a state. The state is an ordered pair (α, β) such that $\alpha^2 + \beta^2 = 1$. If a qubit is in state (α, β) then, when the qubit is measured, the prob that the bit is 0 is α^2 and the prob the bit is 1 is β^2.

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We will elaborate on this on the next slide.

Convention: All Measurements are Local

Convention When we say
Alice measures her qubit in basis θ
we mean
Alice measures her qubit in local basis θ

REMINDER FOR BILL AND EVAN

EVAN: WE MAY WANT TO ELABORATE THIS SINCE WE DON"T KNOW WHAT IT MEANS.
HOWEVER, BY SAYING THIS I NEED NOT SAY LOCAL ALL THE TIME.
IN ANY CASE, we will ask someone who knows stuff about and can discuss *in plain English* rather than *a set of quantum states where each state is primarilty localized to a particular region of space, meaning its wave function is significantly non-zero only within a small defined area and has minimal interaction with distance parts of the system; essentially it represents a way to describe quantum states by focusing on their immediate surroudings, adhering to the principle of locality.*

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So she changes the state of the qubit before measuring it.
This is referred to as **measuring the qubit in a different basis** or in a **different frame.**

COMMENT FROM/FOR EVAN

This is your comment merged with my thoughts.
QUESTIONS FOR PHYSICISTS
The scenario on the prior slide where qubit is in state $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Alice measures it in basis θ. $M_\theta(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\gamma, \delta)$.
Prob of 0 is γ^2 which is NOT $\frac{1}{2}$.
BUT if Alice had an entanlged qubit and she meausres it in basis θ
Prob of 0 IS $\frac{1}{2}$.
Is this really true? If so then explain.

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Next two slides have the first and second coordinate of $M_{\frac{\pi}{6}}(v)$

Example (cont)

First coordinate of $M_{\frac{\pi}{6}}(v)$ is
$\cos(\theta)\alpha - \sin(\theta)\beta = \cos(\frac{\pi}{6})\frac{1}{\sqrt{2}} - \sin(\frac{\pi}{6})\frac{1}{\sqrt{2}}$

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Note $\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2 = \frac{4-2\sqrt{3}}{8} \sim 0.067$

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Second coordinate of $M_{\frac{\pi}{6}}(v)$ is
$\sin(\theta)\alpha + \cos(\theta)\beta = \sin(\frac{\pi}{6})\frac{1}{\sqrt{2}} + \cos(\frac{\pi}{6})\frac{1}{\sqrt{2}}$

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The next few slides investigate this issue further.

How Does θ Affect Pr(0)?

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How Does θ Affect $\Pr(0)$?

Alice has a qubit in state $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
She is going to measure the qubit in bases θ. Let $w = M_\theta(v)$

How Does θ Affect $\Pr(0)$?

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4. For $\frac{3\pi}{4} \le \theta \le \pi$, $\Pr(0)$ goes from 1 to $\frac{1}{2}$.

The next few slides give actual numbers.

$0 \leq \theta \leq \frac{\pi}{4}$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
0	+0.707	+0.707	0.5	0.5
$\pi/60$	+0.669	+0.743	0.448	0.552
$2\pi/60$	+0.629	+0.777	0.396	0.604
$3\pi/60$	+0.588	+0.809	0.345	0.655
$4\pi/60$	+0.545	+0.839	0.297	0.703
$5\pi/60$	+0.500	+0.866	0.250	0.750
$6\pi/60$	+0.454	+0.891	0.206	0.794
$7\pi/60$	+0.407	+0.914	0.165	0.835
$8\pi/60$	+0.358	+0.934	0.128	0.872
$9\pi/60$	+0.309	+0.951	0.095	0.905
$10\pi/60$	+0.259	+0.966	0.067	0.933
$11\pi/60$	+0.208	+0.978	0.043	0.957
$12\pi/60$	+0.156	+0.988	0.024	0.976
$13\pi/60$	+0.105	+0.995	0.011	0.989
$14\pi/60$	+0.052	+0.999	0.003	0.997
$15\pi/60$	+0.000	+1.000	0.000	1.000

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
$15\pi/60$	$+0.000$	$+1.000$	0.000	1.000
$16\pi/60$	-0.052	$+0.999$	0.003	0.997
$17\pi/60$	-0.105	$+0.995$	0.011	0.989
$18\pi/60$	-0.156	$+0.988$	0.024	0.976
$19\pi/60$	-0.208	$+0.978$	0.043	0.957
$20\pi/60$	-0.259	$+0.966$	0.067	0.933
$21\pi/60$	-0.309	$+0.951$	0.095	0.905
$22\pi/60$	-0.358	$+0.934$	0.128	0.872
$23\pi/60$	-0.407	$+0.914$	0.165	0.835
$24\pi/60$	-0.454	$+0.891$	0.206	0.794
$25\pi/60$	-0.500	$+0.866$	0.250	0.750
$26\pi/60$	-0.545	$+0.839$	0.297	0.703
$27\pi/60$	-0.588	$+0.809$	0.345	0.655
$28\pi/60$	-0.629	$+0.777$	0.396	0.604
$29\pi/60$	-0.669	$+0.743$	0.448	0.552
$30\pi/60$	-0.707	$+0.707$	0.500	0.500

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
$30\pi/60$	-0.707	$+0.707$	0.500	0.500
$31\pi/60$	-0.743	$+0.669$	0.552	0.448
$32\pi/60$	-0.777	$+0.629$	0.604	0.396
$33\pi/60$	-0.809	$+0.588$	0.655	0.345
$34\pi/60$	-0.839	$+0.545$	0.703	0.297
$35\pi/60$	-0.866	$+0.500$	0.750	0.250
$36\pi/60$	-0.891	$+0.454$	0.794	0.206
$37\pi/60$	-0.914	$+0.407$	0.835	0.165
$38\pi/60$	-0.934	$+0.358$	0.872	0.128
$39\pi/60$	-0.951	$+0.309$	0.905	0.095
$40\pi/60$	-0.966	$+0.259$	0.933	0.067
$41\pi/60$	-0.978	$+0.208$	0.957	0.043
$42\pi/60$	-0.988	$+0.156$	0.976	0.024
$43\pi/60$	-0.995	$+0.105$	0.989	0.011
$44\pi/60$	-0.999	$+0.052$	0.997	0.003
$45\pi/60$	-1.000	$+0.000$	1.000	0.000

$\frac{3\pi}{4} \leq \theta \leq \pi$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
$45\pi/60$	-1.000	$+0.000$	1.000	0.000
$46\pi/60$	-0.999	-0.052	0.997	0.003
$47\pi/60$	-0.995	-0.105	0.989	0.011
$48\pi/60$	-0.988	-0.156	0.976	0.024
$49\pi/60$	-0.978	-0.208	0.957	0.043
$50\pi/60$	-0.966	-0.259	0.933	0.067
$51\pi/60$	-0.951	-0.309	0.905	0.095
$52\pi/60$	-0.934	-0.358	0.872	0.128
$53\pi/60$	-0.914	-0.407	0.835	0.165
$54\pi/60$	-0.891	-0.454	0.794	0.206
$55\pi/60$	-0.866	-0.500	0.750	0.250
$56\pi/60$	-0.839	-0.545	0.703	0.297
$57\pi/60$	-0.809	-0.588	0.655	0.345
$58\pi/60$	-0.777	-0.629	0.604	0.396
$59\pi/60$	-0.743	-0.669	0.552	0.448
$60\pi/60$	-0.707	-0.707	0.500	0.500

Quantum Bits II: Measure Twice

Exposition by
William Gasarch and Evan Golub

January 7, 2025

Measuring a Qubit Twice In Same Basis

Alice has a qubit. For this scenario both its state and the basis Alice uses are irrelevant.

Scenario 0:

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Upshot If use same basis then they will **agree**.

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Upshot Prob of agreement is $\cos^2(\theta)$.

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Upshot Prob of agreement is $\cos^2(\theta)$.

Note When $\theta = 0$ then Prob of agreement is 1.

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She gets bit b.

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She gets bit b.

2) Bob measures the qubit in basis θ_2,

so in state $w' = M_{\theta_2}(w)$.
The prob that Bob gets b is $\cos^2(\theta_1 - \theta_2)$.

Quantum Bits III: Entanglement

Exposition by
William Gasarch and Evan Golub

January 7, 2025

Alice and Bob Like to Share

We say what Alice and Bob can do if they have qubits that are entangled in a certain way.

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We will only deal with the case where the pair of qubits are an **EPR pair** (EPR stands for Einstein, Podolsky, Rosen) which is the simplest case of Entanglement.

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EPR pairs are also called **Bell Pairs.**

We will define properties of EPR pairs on the next slide.

QUESTIONS FOR EVAN OR Q PERSON

1) We had said that for Alice if she measure qubit, even if she uses a frame θ still gets $(1/2, /1/2)$ and that we would say why later. Why?
Hopefully the answer to why basis matter for quibts that are NOT entangled, this will get answered as well.

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Consider the following two scenarios and tell me if I am correct.

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If I am correct then measuring the same qubit is similar (but not identical) to having entangled bits.
Not Identical since since Alice prob with EPR is $(1/2, 1/2)$ where as with v it can be diff.

EPR Pairs

We will state properties of EPR pairs.

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2) Bob has qubit in state $v_B = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

EPR Pairs

We will state properties of EPR pairs.
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See next slide for how the qubits affect each other.

EPR Pairs Affect Each Other

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Example $\theta_A = \theta_B = 0$. Then Bob will get exactly what Alice got.
Question Does this violate relativity? Alice and Bob are over a light year apart and Alice knows what Bob's Bit is.
Answer No. Alice has no control over what the bit is. If she had control over the bit, or even control over the $\Pr(0)$ and $\Pr(1)$ (e.g., if measuring in a basis changed that) then this would violate Relativity.

Contrast Independent Pairs and EPR pairs

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2) Bob measures his qubit in basis $\frac{\pi}{6}$. State is $M_{\frac{\pi}{6}}(v_B) \sim (0.259, 0.996)$. $0.259^2 \sim 0.0671$. $0.996^2 \sim 0.9329$.

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$\Pr(\text{Bob gets } 1) = \Pr(\text{Alice \& Bob disagree}) = 1 - \cos^2(\frac{\pi}{6} - 0) = 0.25$.

The CHSH Game

Exposition by
William Gasarch and Evan Golub

January 7, 2025

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   If $x \wedge y = 1$ then Alice and Bob win if $a \neq b$.

Classic Strategies

On the next few slides we discuss strategies with an eye towards asking how often they win.

All 0 Strategy

Since $x \wedge y$ is mostly 0, always make $a = b$. So a strong strategy is for Alice and Bob to both send 0. (Both sending a 1 would also be a strong strategy.)

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Alice and Bob win with probability 0.75.

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Next slide analyzes the prob that they win.

Analyzing the Mostly 0 Strategy

x	y	coin	a	b	$x \wedge y$	$a = b$	Wins?
0	0	0	0	0	0	Y	Y
0	0	1	0	0	0	Y	Y
0	1	0	0	0	0	Y	Y
0	1	1	0	0	0	Y	Y
1	0	0	0	0	0	Y	Y
1	0	1	1	0	0	N	N
1	1	0	0	0	1	Y	N
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1	0	0	0	0	0	Y	Y
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In the first four rows the coin flip is irrelevant.

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In the first four rows the coin flip is irrelevant.
If $(x, y) = (1, 0)$ then they win if the coin is 0, so prob $1 - p$.

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2) $(x, y) = (1, 0)$ and the coin is 0. Thats prob $\frac{1}{4} \times (1 - p)$.

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So the prob of winning is $\frac{1}{2} + \frac{1-p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75$.

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So the prob of winning is $\frac{1}{2} + \frac{1-p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75$. No better. Darn!

Is There a Better Strategy?

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1. There is no deterministic strategy that can win with probability more than 0.75.
2. There is no randomized strategy that can win with probability more than 0.75.

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We will show on the next two slides that if Alice and Bob share an EPR pair,

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then Alice and Bob have a strategy that wins the CHSH game with probability $\frac{13}{16} = 0.8125 > 0.75$.

If Alice and Bob Share an EPR Pair . . .

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If Alice and Bob Share an EPR Pair ...

Alice and Bob share an EPR pair.

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Next slide is a strategy for Alice and Bob!

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4. $y = 1$: Bob measures $M_{\frac{\pi}{2}}(v_B)$. b is result.

We analyze all four cases $(x, y) \in \{(0,0), (0,1), (1,0), (1,1)\}$ on the next slides.

Each Scenario

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1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{3}$. Bob: $\frac{\pi}{6}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{3} - \frac{\pi}{6}) = \cos^2(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$.

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1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{3}$. Bob: $\frac{\pi}{6}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{3} - \frac{\pi}{6}) = \cos^2(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$.
2. $(x, y) = (0, 1)$. Alice: $\frac{\pi}{3}$. Bob: $\frac{\pi}{2}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{3} - \frac{\pi}{2}) = \cos^2(-\frac{\pi}{6}) = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$.

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3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{6}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{6} - 0) = \cos^2(\frac{\pi}{6}) = \frac{3}{4}$.

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Hence the prob of a win is
$\frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times 1 = \frac{13}{16} = 0.8125$.

What Does This Mean?

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1. Physicists have actually done this in the lab.
2. This is evidence that quantum mechanics is correct.
3. There are things we can do **better** in the quantum world than in the classical world.

CSHS: If Alice and Bob Share an EPR pair Can they do better than 0.8125?

Exposition by
William Gasarch and Evan Golub

January 7, 2025

Can We Do Better?

We have shown the following:

If Alice and Bob share an EPR pair then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125

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Assume Alice and Bob share an EPR pair.

Vote Which of the following is true:

Can We Do Better?

We have shown the following:

If Alice and Bob share an EPR pair then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125

Assume Alice and Bob share an EPR pair.

Vote Which of the following is true:

1. Alice and Bob have a strategy that wins the CHSH game with Prob $p > 0.8125$ and this is **known**.

Can We Do Better?

We have shown the following:

If Alice and Bob share an EPR pair then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125

Assume Alice and Bob share an EPR pair.

Vote Which of the following is true:

1. Alice and Bob have a strategy that wins the CHSH game with Prob $p > 0.8125$ and this is **known**.
2. The best Alice and ... can do is 0.8125 and this is **known**.

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If Alice and Bob share an EPR pair then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125
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3. The question of if Alice and Bob can do better than 0.8125 is **Unknown to Science**.

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If Alice and Bob share an EPR pair then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125

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Answer on the next slide.

Alice and Bob Can Do Better than 0.8125

Alice and Bob share an EPR pair.

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Alice and Bob share an EPR pair.
Alice gets x, Bob gets y.

Alice and Bob Can Do Better than 0.8125

Alice and Bob share an EPR pair.
Alice gets x, Bob gets y.

1. $x = 0$: Alice measures $M_{\frac{\pi}{4}}(v_A)$. a is result.

Alice and Bob Can Do Better than 0.8125

Alice and Bob share an EPR pair.
Alice gets x, Bob gets y.

1. $x = 0$: Alice measures $M_{\frac{\pi}{4}}(v_A)$. a is result.
2. $x = 1$: Alice measures v_A in the standard basis. a is result.

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3. $y = 0$: Bob measures $M_{\frac{\pi}{8}}(v_B)$. b is result.

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We analyze all four cases $(x, y) \in \{(0,0),(0,1),(1,0),(1,1)\}$ on the next slides.

Each Scenario

Alice and Bob share an EPR pair.

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1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{4} - \frac{\pi}{8}) = \cos^2(\frac{\pi}{8}) \sim 0.853$.

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2. $(x, y) = (0, 1)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{3\pi}{8}$. Prob that $a = b$ is $\cos^2(\frac{3\pi}{8} - \frac{\pi}{4}) = \cos^2(\frac{\pi}{8}) \sim 0.853$.

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3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{8}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{8} - 0) = \cos^2(\frac{\pi}{8}) \sim 0.853$.

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1. $(x,y)=(0,0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob that $a=b$ is $\cos^2(\frac{\pi}{4}-\frac{\pi}{8})=\cos^2(\frac{\pi}{8})\sim 0.853$.
2. $(x,y)=(0,1)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{3\pi}{8}$. Prob that $a=b$ is $\cos^2(\frac{3\pi}{8}-\frac{\pi}{4})=\cos^2(\frac{\pi}{8})\sim 0.853$.
3. $(x,y)=(1,0)$. Alice: 0. Bob: $\frac{\pi}{8}$. Prob that $a=b$ is $\cos^2(\frac{\pi}{8}-0)=\cos^2(\frac{\pi}{8})\sim 0.853$.
4. $(x,y)=(1,1)$. Alice: 0. Bob: $3\pi/8$. Prob that $a\neq b$ is $1-\cos^2(\frac{3\pi}{8}-0)=1-\cos^2(\frac{3\pi}{8})=1-(1-\cos^2(\frac{\pi}{8}))=\cos^2(\frac{\pi}{8})\sim 0.853$.

Hence the prob of a win is
$\frac{3}{4}(0.853)+\frac{1}{4}(0.853)=0.853>0.8125$.

Each Scenario

Alice and Bob share an EPR pair.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob that $a = b$ is $\cos^2(\frac{\pi}{4} - \frac{\pi}{8}) = \cos^2(\frac{\pi}{8}) \sim 0.853$.
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Hence the prob of a win is
$\frac{3}{4}(0.853) + \frac{1}{4}(0.853) = 0.853 > 0.8125$.
The exact prob of winning is $\cos^2(\frac{\pi}{8})$.

COMMENT FOR EVAN

The next few slides were in the old version.
After that there are around THIRTY new slides.
Enjoy reading!

CSHS: If Alice and Bob Share an EPR pair Can they do better than ~ 0.854?

Exposition by
William Gasarch and Evan Golub

January 7, 2025

Can Alice and Bob Do Better?

Assume Alice and Bob share an EPR pair.

Can Alice and Bob Do Better?

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Vote Which of the following is true:

Can Alice and Bob Do Better?

Assume Alice and Bob share an EPR pair.

Vote Which of the following is true:

1. Alice and Bob have a strategy that wins the CHSH game with Prob $p > \cos^2(\frac{\pi}{8})$ (approx 0.853) and this is **known**.

Can Alice and Bob Do Better?

Assume Alice and Bob share an EPR pair.

Vote Which of the following is true:

1. Alice and Bob have a strategy that wins the CHSH game with Prob $p > \cos^2(\frac{\pi}{8})$ (approx 0.853) and this is **known**.
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1. Alice and Bob have a strategy that wins the CHSH game with Prob $p > \cos^2(\frac{\pi}{8})$ (approx 0.853) and this is **known**.
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3. The question of if Alice and Bob can do better than $\cos^2(\frac{\pi}{8})$ is **Unknown to Science**.

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Answer to a particular part of this problem on the Next Page.

What If Alice & Bob Use Diff Angles?

Recall

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1. First Strategy:
   Alice: If $x = 0$ then use $\frac{\pi}{3}$. If $x = 1$ then use 0.

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   Alice: If $x = 0$ then use $\frac{\pi}{3}$. If $x = 1$ then use 0.
   Bob: If $y = 0$ then use $\frac{\pi}{6}$. If $y = 1$ then use $\frac{\pi}{2}$.

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   This yielded prob of winning 0.8125.

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   Alice: If $x = 0$ then use $\frac{\pi}{4}$. If $x = 1$ then use 0.

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What If Alice & Bob Use Diff Angles?

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1. First Strategy:
   Alice: If $x = 0$ then use $\frac{\pi}{3}$. If $x = 1$ then use 0.
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Can Alice and Bob obtain a higher prob of winning with a different choice of angles?

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1. First Strategy:
   Alice: If $x = 0$ then use $\frac{\pi}{3}$. If $x = 1$ then use 0.
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   This yields prob of winning $\cos^2(\frac{\pi}{8}) \sim 0.853$.

Can Alice and Bob obtain a higher prob of winning with a different choice of angles?
Answer in the next 30 or so slides.

Protocol with Parameters

Alice and Bob share an EPR pair.

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Alice and Bob share an EPR pair.
The angles they use will be $\theta(A,0)$, $\theta(A,1)$, $\theta(B,0)$, $\theta(B,1)$.

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Alice gets x, Bob gets y.

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Alice gets x, Bob gets y.

1. $x = 0$: Alice measures $M_{\theta(A,0)}(v_A)$. a is result.

Protocol with Parameters

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Alice and Bob share an EPR pair.
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Protocol with Parameters

Alice and Bob share an EPR pair.
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3. $y = 0$: Bob measures $M_{\theta(B,0)}(v_B)$. b is result.
4. $y = 1$: Bob measures $M_{\theta(B,1)}(v_B)$. b is result.

We analyze all four cases $(x,y) \in \{(0,0),(0,1),(1,0),(1,1)\}$ on the next slides.

Each Scenario

Alice and Bob share an EPR pair.

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Alice and Bob share an EPR pair.

1. $(x, y) = (0, 0)$. Alice: $\theta(A, 0)$. Bob: $\theta(B, 0)$.
   Prob$(a = b)$=$\cos^2(\theta(A, 0) - \theta(B, 0))$.

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4. $(x, y) = (1, 1)$. Alice: $\theta(A, 1)$. Bob: $\theta(B, 1)$.
   Prob$(a \neq b)$=$1 - \cos^2(\theta(A, 1) - \theta(B, 1))$.

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4. $(x, y) = (1, 1)$. Alice: $\theta(A, 1)$. Bob: $\theta(B, 1)$.
   Prob$(a \neq b)$=$1 - \cos^2(\theta(A, 1) - \theta(B, 1))$.

Hence the prob of a win is

Each Scenario

Alice and Bob share an EPR pair.

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$$\frac{1}{4}(\cos^2(\theta(A,0)-\theta(B,0))+\cos^2(\theta(A,0)-\theta(B,1))+\cos^2(\theta(A,1)-\theta(B,0))+(1-\cos^2(\theta(A,1)-\theta(B,1))))$$

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2) There may be some way to show that 0.853 is the max value without a program but we are not going to go in that direction.

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However In 1980 Tsirelson proved the following:

Even allowing Alice and Bob to share many EPR pairs, there is no strategy that gives a prob of winning $> \cos^2(\frac{\pi}{8})$.

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Final Thoughts

1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.
2. If Alice and Bob share an EPR pair then there is a strategy that has prob of winning $\cos^2(\frac{\pi}{8}) \sim 0.853$.
3. I am amazed that with a shared EPR pair Alice and Bob can do better.
4. I am amazed that with a shared EPR pair Alice and Bob can do **so much better**. I would have have thought something like $0.75 + \epsilon$.
5. Even with many EPR pairs and any kind of strategy Alice and Bob cannot do better than $\cos^2(\frac{\pi}{8})$. I am not amazed this is true, but I am amazed its been proven. (Proof is hard.)