0.1 Robyn and Bob Play Tic-Tac-Toe While Drunk

Robyn and Bob enjoy playing tic-tac-toe. In the spirit of Ramsey Theory, (1) they are using R and B instead of X and O, they think of the game as coloring points in the grid. As is well known, if they both play perfectly, the game will be a tie. Clearly, if they both play drunk, the game can be a tie. We make this statement more rigorous and change it a little.

1. Let \( A = \{(x, y) : 1 \leq x, y \leq 3\} \).

2. A move in Tic-Tac-Toe is to take a point of \( A \) that is not colored and give it your color.

Normally if a player colors any of the following 3-point-sets all the same then they win:

\[
\begin{align*}
(1, 1), (1, 2), (1, 3) \\
(1, 1), (2, 1), (3, 1) \\
(1, 1), (2, 2), (3, 3) \\
(2, 1), (2, 2), (2, 3) \\
(3, 1), (2, 2), (1, 3) \text{ (This one is different! We’ll see why soon.)} \\
(3, 1), (2, 2), (1, 3) \\
(1, 2), (2, 2), (3, 2) \\
(1, 3), (2, 3), (3, 3)
\end{align*}
\]

We will consider a slightly different game. For 7 of these we get that, for each position, the number either stays constant or goes up. For example in

\[
(1, 1), (1, 2), (1, 3)
\]

the number in the first coordinate is constant and the number in the second coordinate goes up. There is one exception. For

\[
(3, 1), (2, 2), (1, 3)
\]

the number in the first coordinate goes up and the number in the second coordinate goes down. We will not count this as a line.

**Def 0.1.1** Let \( n, t \in \mathbb{N} \).
1. \( C_n^t = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in [t]\} \). Note that \( C_2^2 \) is the board for the usual tic-tac-toe game. We call \( n \) the dimension and \( t \) the size.

2. A line in \( C_n^t \) is a sequence of \( t \) points \( p_1, \ldots, p_t \in C_n^t \) such that, for all \( 1 \leq i \leq n \) either
   
   \( (a) \ p_1(i) = \cdots = p_t(i), \) or
   \( (b) \ p_1(i) = 1, \ p_2(i) = 2, \ldots, p_t(i) = t. \)

3. 2-player Tic-Tac-Toe on \( C_n^t \) is the following game. Robyn and Bob alternate picking uncolored points and coloring them. Robyn (Bob) uses color R (B). Robyn goes first. The first player to get a monochromatic line in their color wins.

4. Henceforth we will use the term \( \text{mono line}. \)

5. Let \( c \in \mathbb{N} \). \( c \)-player Tic-Tac-Toe on \( C_n^t \) is the following game. Players \( A_1, \ldots, A_c \) go in the order \( A_1, \ldots, A_c, A_1, \ldots, A_c, \ldots \) picking uncolored points and coloring them. \( A_i \) uses color \( i. \) The first player to get a mono line in their color wins.

Now back to Robyn and Bob. Clearly there is a way they could play so that the game is a tie. We want to find them a board \( C_n^t \) so that no matter how badly they play someone wins. We will phrase this as

\( \text{for which } n, t \text{ is it the case that, for all 2-colorings } \chi \text{ of } C_n^t, \text{ there exists a mono line.} \)

If the size of the board is 2, this is easy.

**Exercise 1** Show that for all 2-colorings \( \chi \) of \( C_2^2 \) there is a mono line.

What if the size of the board is 3? Alas, there is a coloring where nobody wins.

**Theorem 0.1.2** Let \( t \geq 3. \) There exists a 2-coloring of \( C_2^t \) that has no mono lines. The 2-coloring will have half R and half B (or B − 1 if \( t \) is odd) so the coloring could arise in a drunk game.
0.1. ROBYN AND BOB PLAY TIC-TAC-TOE WHILE DRUNK

Proof sketch:

We do an example for \( t = 4 \) which will show how to do this for any even \( t \).

\[
\begin{array}{cccc}
R & R & B & B \\
B & B & R & R \\
R & R & B & B \\
B & B & R & R \\
\end{array}
\]

We leave the case of \( t \) odd to the reader.

So what to do? What if we have Robyn and Bob play on a cube? Or a hypercube? We will show that if Robyn and Bob play on an 8-dimensional hypercube with sides of length 3 then someone must win.

**Theorem 0.1.3** For all 2-colorings of \( C^3_8 \) there exists a mono line.

**Proof:**

Let \( \chi \) be a 2-coloring of \( C^3_8 \). The colors are R and B. Assume, by way of contradiction, that there are no mono lines.

We look points of \( C^3_6 \) and append 2 more coordinates, either 11, 12, or 22. For example, we associate to 221311 the set

\[
A = \{22131111, 22131112, 22131122\}.
\]

Since \( A \) has 3 points and \( \chi \) maps to 2 colors, 2 of the points must be colored the same. The lack of mono lines will force another points color. For example If \( \chi(22131111) = R \) and \( \chi(22131122) = R \) then \( \chi(22131112) = B \).

More generally, for every \( abcdef \in C^3_6 \) at least one of the following 6 scenarios holds.

1. \( \chi(abcdef11) = \chi(abcdef12) = R \) so \( \chi(abcdef13) = B \).
2. \( \chi(abcdef11) = \chi(abcdef22) = R \) so \( \chi(abcdef33) = B \).
3. \( \chi(abcdef12) = \chi(abcdef22) = R \) so \( \chi(abcdef32) = B \).
4. \( \chi(abcdef11) = \chi(abcdef12) = B \) so \( \chi(abcdef13) = R \).
5. \( \chi(abcdef11) = \chi(abcdef22) = B \) so \( \chi(abcdef33) = R \).
6. \( \chi(abcdef12) = \chi(abcdef22) = B \) so \( \chi(abcdef32) = R \).
Let $\chi' : C_6^3 \to [6]$ map each point of $C_6^3$ to the smallest index in the above list that describes a scenario for that point.
Look at the following 7 points of $C_6^3$:

$$B = \{111111, 111112, 111122, 111222, 112222, 122222, 222222\}$$

Since $B$ has 7 elements and $\chi'$ maps to 6 colors, two of the points map to the same color, so the same scenario. We do an indicative example and leave the general proof to the reader.

Suppose for instance $111122$ and $122222$ fall into class (3), thus

1. $\chi(11112212) = \chi(11112222) = R$ and $\chi(11112232) = B$.
2. $\chi(12222212) = \chi(12222222) = R$ and $\chi(12222232) = B$.

Since $\chi(11112232) = \chi(12222232) = B$ and there are no mono lines, $\chi(13332232) = R$.

But then

$$\chi(11112212) = \chi(12222222) = \chi(13332232) = R.$$

This contradicts that there are no mono lines.

Theorem ?? is not optimal. Hindman & Tressler [?] proved the following.

**Theorem 0.1.4**

For all 2-colorings of $C_3^3$ there exists a mono line.

2. There exists a 2-coloring of $C_3^3$ without a mono line. (The coloring has 14 R’s and 13 B’s so it could arise in a drunk game.)

### 0.2 $A_1, \ldots, A_c$ Play Tic-Tac-Toe While Drunk

What if there are $c$ players? Now the question is

for which $n, t$ is it the case that, for all $c$-colorings $\chi$ of $C_n^t$, there exists a mono line.

The case of $t = 2$ is easy.
Theorem 0.2.1 Let $c \in \mathbb{N}$, $c \geq 2$. For all $c$-colorings $\chi : C^2_c \to [c]$ there exists a mono line.

Proof: Consider the points
- $R \cdots R$ ($n$ R’s and 0 B’s)
- $R \cdots RB$ ($n - 1$ R’s and 1 B’s)
- $R \cdots RBB$ ($n - 2$ R’s and 2 B’s)
- $\vdots$
- $B \cdots B$ (0 R’s and $n$ B’s)

Since $\chi$ uses $c$ colors and there are $c + 1$ points, two of the points have the same color. Those two points form a mono line. 

BILL TO BILL: WHAT HAPPENS WITH $\chi : C^2_{c-1} \to [c]$. 