Hat Problem: People Standing in a Line

William Gasarch-U of MD
**The Set Up**

100 people working together as a team, must stand in a line. Each person can see the heads of everyone in front of her, but not her own head, or the heads of those in back of her. BEFORE hats are placed (the next step) they can discuss strategy; however, the adversary listens in on that conversation.
**The Set Up**

100 people working together as a team, must stand in a line. Each person can see the heads of everyone in front of her, but not her own head, or the heads of those in back of her. BEFORE hats are placed (the next step) they can discuss strategy; however, the adversary listens in on that conversation.

**The Adversary’s Move:** The Adversary places either a red hat or a blue hat on top of each contestant’s head. The contestants cannot communicate at all except as specified in the next step.
The Set Up

100 people working together as a team, must stand in a line. Each person can see the heads of everyone in front of her, but not her own head, or the heads of those in back of her. BEFORE hats are placed (the next step) they can discuss strategy; however, the adversary listens in on that conversation.

The Adversary’s Move: The Adversary places either a red hat or a blue hat on top of each contestant’s head. The contestants cannot communicate at all except as specified in the next step.

The Contestants Move: After the hats have been placed, each contestant, in turn starting from the back of the line and proceeding one by one to the front of the line, will call out one of the two colors, red or blue. Their goal is to get as many people as possible to correctly call out their own hat color.
They CAN Get $\geq n/2$ Correct

The people are in a line $p_1, p_2, p_3, \cdots p_n$. 
They CAN Get $\geq n/2$ Correct

The people are in a line $p_1, p_2, p_3, \cdots p_n$.

1. $p_1$ says the majority color. They all say that color.
They CAN Get $\geq n/2$ Correct

The people are in a line $p_1, p_2, p_3, \ldots p_n$.

1. $p_1$ says the majority color. They all say that color. $n/2$. 
The people are in a line $p_1, p_2, p_3, \cdots p_n$.

1. $p_1$ says the majority color. They all say that color. $n/2$.
2. For all $1 \leq i \leq n/2$
   
   $p_{2i+1}$ says $p_{2i+2}$’s color. $p_{2i+2}$ says her color.
They CAN Get $\geq n/2$ Correct

The people are in a line $p_1, p_2, p_3, \ldots p_n$.

1. $p_1$ says the majority color. They all say that color. $n/2$.
2. For all $1 \leq i \leq n/2$
   $p_{2i+1}$ says $p_{2i+2}$’s color. $p_{2i+2}$ says her color. $n/2$. 
Work on the Following in Groups

$n$ people. 2 hat colors:

1. Is there a strategy that is guaranteed to get MORE THAN $n/2$ hats correct?
2. What is the best they can do?
3. If finish early work on 3 colors, 4 colors, etc.
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. For all $1 \leq i \leq n/3$
   
   $p_{3i}$ says $R$ if $p_{3i+1}, p_{3i+2}$ are same, $B$ otherwise. $p_{3i+1}$ can deduce his color, then $p_{3i+2}$ can deduce her color.
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. For all $1 \leq i \leq n/3$

   $p_{3i}$ says $R$ if $p_{3i+1}, p_{3i+2}$ are same, $B$ otherwise. $p_{3i+1}$ can deduce his color, then $p_{3i+2}$ can deduce her color. $2n/3$.
\( n \) people, 2 Hat Colors, Several Answers

\( p_i \) is person \( i \).

1. For all \( 1 \leq i \leq n/3 \)

   \( p_{3i} \) says \( \text{R} \) if \( p_{3i+1}, p_{3i+2} \) are same, \( \text{B} \) otherwise. \( p_{3i+1} \) can deduce his color, then \( p_{3i+2} \) can deduce her color. \( 2n/3 \).

2. \( p_1, p_2, \ldots, p_{\lfloor \log_2 n \rfloor} \) spell out in binary the number of \( \text{red} \) hats among \( p_{\lfloor \log_2 n \rfloor + 1}, \ldots, p_n \). Each person can deduce their color based on the number and the prior utterances.
\( n \) people, 2 Hat Colors, Several Answers

\( p_i \) is person \( i \).

1. For all \( 1 \leq i \leq n/3 \)
   
   \( p_{3i} \) says \( R \) if \( p_{3i+1}, p_{3i+2} \) are same, \( B \) otherwise. \( p_{3i+1} \) can deduce his color, then \( p_{3i+2} \) can deduce her color. \( 2n/3 \).

2. \( p_1, p_2, \ldots, p_{\log_2 n} \) spell out in binary the number of red hats among \( p_{\log_2 n+1}, \ldots, p_n \). Each person can deduce their color based on the number and the prior utterances. \( n - \log(n) \).
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. For all $1 \leq i \leq n/3$
   $p_{3i}$ says $R$ if $p_{3i+1}, p_{3i+2}$ are same, $B$ otherwise. $p_{3i+1}$ can deduce his color, then $p_{3i+2}$ can deduce her color. $2n/3$.

2. $p_1, p_2, \ldots, p_{\log_2 n}$ spell out in binary the number of red hats among $p_{\log_2 n+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \log(n)$.

3. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances.
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. For all $1 \leq i \leq n/3$
   $p_{3i}$ says $R$ if $p_{3i+1}, p_{3i+2}$ are same, $B$ otherwise. $p_{3i+1}$ can deduce his color, then $p_{3i+2}$ can deduce her color. $2n/3$.

2. $p_1, p_2, \ldots, p_{\lfloor \log_2 n \rfloor}$ spell out in binary the number of red hats among $p_{\lfloor \log_2 n+1 \rfloor}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \log(n)$.

3. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances. $n - 1$.  

Optimal!
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. For all $1 \leq i \leq n/3$
   
   $p_{3i}$ says R if $p_{3i+1}, p_{3i+2}$ are same, B otherwise. $p_{3i+1}$ can deduce his color, then $p_{3i+2}$ can deduce her color. $2n/3$.

2. $p_1, p_2, \ldots, p_{\log_2 n}$ spell out in binary the number of red hats among $p_{\log_2 n+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \log_2(n)$.

3. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances. $n - 1$. Optimal!
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. For all $1 \leq i \leq n/3$
   $p_{3i}$ says $R$ if $p_{3i+1}, p_{3i+2}$ are same, $B$ otherwise. $p_{3i+1}$ can deduce his color, then $p_{3i+2}$ can deduce her color. $2n/3$.

2. $p_1, p_2, \ldots, p_{\lg n}$ spell out in binary the number of red hats among $p_{\lg n+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \lg(n)$.

3. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances. $n - 1$. Optimal!

4. BILL- TELL the Story!
More Hat Colors!

What if $n$ people, 3 hats colors? 4 ? $c$?

(If you finish early than look at an infinite number of people and 2 hat colors.)
\( n \) people, 3 Hat Colors Answer

\( p_i \) is person \( i \).
$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: red if the numb of reds is even, blue otherwise.
$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: red if the numb of reds is even, blue otherwise

Rephrase. red is 0, blue is 1, $h_i$ is hat on $p_i$.

$$p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{2}$$
n people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: red if the numb of reds is even, blue otherwise.

Rephrase. red is 0, blue is 1, $h_i$ is hat on $p_i$.

\[ p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{2} \]

For 3 colors:

\[ p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{3} \]
$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: red if the numb of reds is even, blue otherwise

Rephrase. red is 0, blue is 1, $h_i$ is hat on $p_i$.

\[
p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{2}
\]

For 3 colors:

\[
p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{3}
\]

Let $s_j$ be what $p_j$ says. $p_i$ can deduce that

\[
h_i \equiv s_1 - \sum_{j=2}^{i-1} s_j \pmod{3}
\]
Infinite Number of People!

Infinite number of people and 2 colors of hats.

Want a protocol such that all but a finite number get it right.
People are \( p_1, p_2, \ldots \).
People are $p_1, p_2, \ldots$
They meet ahead of time. Let $H = \{R, B\}^\omega$. 
People are $p_1, p_2, \ldots$
They meet ahead of time. Let $H = \{R, B\}^\omega$.
They define

\[ x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often} \]
People are $p_1, p_2, \ldots$
They meet ahead of time. Let $H = \{R, B\}^{\omega}$.
They define
\[ x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often} \]
$\equiv$ is an equiv rel, so a partition. Every $x \in H$ is in one part.
Infinite Number of People! 2 Hat Colors

People are \( p_1, p_2, \ldots \).

They meet ahead of time. Let \( H = \{R, B\}^\omega \).

They define

\[ x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often} \]

\( \equiv \) is an equiv rel, so a partition. Every \( x \in H \) is in one part.

1. (Preprocess) \( p_i \)'s pick a REPRESENTATIVE from each part.
People are $p_1, p_2, \ldots$.

They meet ahead of time. Let $H = \{R, B\}^\omega$.

They define

$$x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often}$$

$\equiv$ is an equiv rel, so a partition. Every $x \in H$ is in one part.

1. (Preprocess) $p_i$’s pick a REPRESENTATIVE from each part.

2. Each $p_i$ sees all but a finite number of hats. So they know which part they are in. Call representative of the part, REP.
People are $p_1, p_2, \ldots$.
They meet ahead of time. Let $H = \{R, B\}^\omega$.
They define
\[ x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often} \]
$\equiv$ is an equiv rel, so a partition. Every $x \in H$ is in one part.

1. (Preprocess) $p_i$’s pick a REPRESENTATIVE from each part.
2. Each $p_i$ sees all but a finite number of hats. So they know which part they are in. Call representative of the part, REP.
3. Each $p_i$ says the color in the $i$th position in REP.

They all end up collectively saying REP, which is only a finite number of hats away from the real answer.
People are $p_1, p_2, \ldots$
They meet ahead of time. Let $H = \{R, B\}^\omega$.
They define
\[ x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often} \]
\[ \equiv \text{ is an equiv rel, so a partition. Every } x \in H \text{ is in one part.} \]
1. (Preprocess) $p_i$’s pick a REPRESENTATIVE from each part.
2. Each $p_i$ sees all but a finite number of hats. So they know which part they are in. Call representative of the part, REP.
3. Each $p_i$ says the color in the $i$th position in REP.
They all end up collectively saying REP, which is only a finite number of hats away from the real answer.
Can They Do Better?

Vote

1. There is a protocol and a constant \( C \) so that the protocol always results in \( \leq C \) hats wrong, and this is known.

2. For all protocols and all constant \( C \) there is a way for the adversary to put hats on peoples heads so that the protocol gets \( \geq C \) wrong, and this is known.

3. The question
   
   Is there a protocol and a \( C \) such that BLAH BLAH is independent of ZFC.

4. Which of 1,2, or 3 happens is **Unknown to Science**.
Can They Do Better?

Vote

1. There is a protocol and a constant $C$ so that the protocol always results in $\leq C$ hats wrong, and this is known.

2. For all protocols and all constant $C$ there is a way for the adversary to put hats on peoples heads so that the protocol gets $\geq C$ wrong, and this is known.

3. The question
   Is there a protocol and a $C$ such that BLAH BLAH is independent of ZFC.

4. Which of 1,2, or 3 happens is **Unknown to Science**.

Work on it in small groups.
Protocol that gets $\leq 1$ Wrong

1. $p_1$ determines REP. He says:
Protocol that gets $\leq 1$ Wrong

1. $p_1$ determines REP. He says:
   - $R$ if REP and $h_2, \ldots$ Differ In An Odd Number of Places
Protocol that gets $\leq 1$ Wrong

1. $p_1$ determines REP. He says:
   - **R** if REP and $h_2, \ldots$ Differ In An Odd Number of Places
   - **B** if REP and $h_2, \ldots$ Differ In An Even Number of Places
Protocol that gets $\leq 1$ Wrong

1. $p_1$ determines REP. He says:
   - $R$ if REP and $h_2, \ldots$ Differ In An Odd Number of Places
   - $B$ if REP and $h_2, \ldots$ Differ In An Even Number of Places

2. $p_2$ knows parity of how much $h_2, \ldots$, differs from REP
   (From what $p_1$ said)

   $p_2$ knows parity of how much $h_3, \ldots$, differs from REP
   (She sees)

   hence she can deduce $h_2$. 
Protocol that gets \( \leq 1 \) Wrong

1. \( p_1 \) determines REP. He says:
   - \( R \) if REP and \( h_2, \ldots \) Differ In An Odd Number of Places
   - \( B \) if REP and \( h_2, \ldots \) Differ In An Even Number of Places

2. \( p_2 \) knows parity of how much \( h_2, \ldots \), differs from REP
   (From what \( p_1 \) said)

   \( p_2 \) knows parity of how much \( h_3, \ldots \), differs from REP
   (She sees)

   hence she can deduce \( h_2 \).

3. Similar for all \( p_i \) with \( i \geq 2 \).
1. $p_1$ determines REP. He says:
   \[\text{R if REP and } h_2, \ldots \text{ Differ In An Odd Number of Places}\]
   \[\text{B if REP and } h_2, \ldots \text{ Differ In An Even Number of Places}\]

2. $p_2$ knows parity of how much $h_2, \ldots$, differs from REP
   (From what $p_1$ said)

   $p_2$ knows parity of how much $h_3, \ldots$, differs from REP
   (She sees)

   hence she can deduce $h_2$.

3. Similar for all $p_i$ with $i \geq 2$.

The only one who might get it wrong is $p_1$. 