NIM Games for Tyler and Noei

William Gasarch-U of MD
Tyler and Noei play a game:
Tyler and Noei play a game:

1. There are 7 mints on the table.
Tyler and Noei play a game:

1. There are \textbf{7 mints} on the table.
2. Tyler removes \textbf{1,2, OR 3} mints.

They keep doing this.

The person who removes the last mint wins.

Play the game and see if one of the players can always win.
Tyler and Noei play a game:
1. There are 7 mints on the table.
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Tyler and Noei play a game:

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4. Tyler then removes 1, 2, OR 3 mint.
Tyler and Noei play a game:

1. There are **7 mints** on the table.
2. Tyler removes **1,2, OR 3** mints.
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5. They keep doing this.
7 Mints, 1-2-3 Moves, Game

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Play the game and see if one of the players can always win.
Tyler Can Always Win

1. Tyler removes 3 mints. There are now 4 mints.

2. 2.1 If Noei removes 1 mint, Tyler removes 3. Tyler wins.
    2.2 If Noei removes 2 mints, Tyler removes 2. Tyler wins.
    2.3 If Noei removes 3 mints, Tyler removes 1. Tyler wins.
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Work on:

1. Can Tyler always win if the game begins with 5 mints?
2. Can Tyler always win if the game begins with 6 mints?
3. Can Tyler always win if the game begins with 7 mints? (Yes)
4. Can Tyler always win if the game begins with 8 mints?
Work on:

1. Can Tyler always win if the game begins with 5 mints?
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1. Can Tyler always win if the game begins with 5 mints?
   Tyler removes 1, there are now 4. Same as before.
2. Can Tyler always win if the game begins with 6 mints?
What about 5,6,7,8 Mints? Answers

Work on:

1. Can Tyler always win if the game begins with 5 mints? Tyler removes 1, there are now 4. Same as before.
2. Can Tyler always win if the game begins with 6 mints? Tyler removes 2, there are now 4. Same as before.
What about 5, 6, 7, 8 Mints? Answers

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1. Can Tyler always win if the game begins with 5 mints? Tyler removes 1, there are now 4. Same as before.
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3. Can Tyler always win if the game begins with 7 mints? (Yes)
What about 5, 6, 7, 8 Mints? Answers

Work on:

1. Can Tyler always win if the game begins with 5 mints? Tyler removes 1, there are now 4. Same as before.
2. Can Tyler always win if the game begins with 6 mints? Tyler removes 2, there are now 4. Same as before.
3. Can Tyler always win if the game begins with 7 mints? (Yes) Tyler removes 3, there are now 4. Same as before.
What about $5,6,7,8$ Mints? Answers

Work on:

1. Can Tyler always win if the game begins with $5$ mints? Tyler removes 1, there are now 4. Same as before.
2. Can Tyler always win if the game begins with $6$ mints? Tyler removes 2, there are now 4. Same as before.
3. Can Tyler always win if the game begins with $7$ mints? (Yes) Tyler removes 3, there are now 4. Same as before.
What about 8 Mints? Answers

1. If Tyler removes 1 then Noei removes 3. Now Tyler is looking at 4 mints and Noei can win.
2. If Tyler removes 2 then Noei removes 2. Now Tyler is looking at 4 mints and Noei can win.
3. If Tyler removes 3 then Noei removes 1. Now Tyler is looking at 4 mints and Noei can win.
What about 8 Mints? Answers

Noei Wins!
What about 8 Mints? Answers

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Work on:

1. Can Tyler always win if the game begins with 0 mints?
What about 1,2,3,4 Mints?

Work on:

1. Can Tyler always win if the game begins with 0 mints? No, he can’t move.
2. Can Tyler always win if the game begins with 1 mints? Yes—remove 1.
3. Can Tyler always win if the game begins with 2 mints? Yes—remove 2.
4. Can Tyler always win if the game begins with 3 mints? Yes—remove 3.
5. Can Tyler always win if the game begins with 4 mints? No; whoever wins whatever. Tyler does, Noei can remove the rest of the mints.
What about 1,2,3,4 Mints?

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Table of Who Wins

I means player I. Has been Tyler.
II means player II. Has been Noei.
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What is the pattern?
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What is the pattern?

Player I wins if the Numb of Mints IS NOT divisible by 4.
Player II wins if the Numb of Mints IS divisible by 4.
1-2-3-4 Moves, Game

Player I and II play a game:
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1. There are \( n \) mints on the table.
Player I and II play a game:

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6. The person who removes the last mint wins.

Play the game and see if one of the players can always win. Try to figure out:
- When Player I wins.
- When Player II wins.
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1-4-5 Game

1. 0 mints: II wins – I can’t move.
2. 1 mint: I wins – Remove 1
3. 2 mints: II wins – I cannot get to 0.
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6. 5 mints: I wins – Remove 5.
8. 7 mints: I wins – Remove 5.
9. 8 mints: II wins – Cannot get to 0 or 2.

We want a way to describe this pattern.
1-4-5 Game

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1-4-5 Game

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We want a way to describe this pattern.
For 1-2-3 Game we said

- Player I wins if $n$ is not divisible by 4
- Player II wins if $n$ is divisible by 4.
Notation

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\text{If } n \equiv 0 \pmod{4} \text{ pronounced } "n \text{ is congruent to 0 mod 4}" \]

Means that \( n \) is divisible by 4.
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Means that $n$ is divisible by 4.

- Player I wins if $n \not\equiv 0 \pmod{4}$
- Player II wins if $n \equiv 0 \pmod{4}$
For 1-3-4 Game we said

- Player I wins if when $n$ is divisible by 7 get a remainder that is NOT 0 or 2.
- Player II wins if when divide $n$ by 7 get a remainder of 0 or 2.

We have a notation for this:

$n \equiv 0 \pmod{7}$

Means that $n$ is divisible by 7.

$n \equiv 2 \pmod{7}$

Means that when you divide $n$ by 7 the remainder is 2.
For 1-3-4 Game we said

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- Player I wins if \( n \equiv 1, 3, 5, 6 \pmod{7} \).
- Player II wins if \( n \equiv 0, 2 \pmod{7} \)
Player I wins if \( n \equiv 1, 3, 4, 5, 6, 7 \pmod{8} \).

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- Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.
- Player II wins if $n \equiv 0, 2 \pmod{8}$. 

1-4-5
Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.

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Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$. 
## 1-4-5

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- **Player I wins if** $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.
- **Player II wins if**
Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.
Player II wins if $n \equiv 0, 2 \pmod{8}$.
Patterns of Patterns!

Recall:

1. 1-2-3 Game:
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   - Player II wins if $n \equiv 0 \pmod{4}$.

2. 1-2-3-4 Game:
   - Player I wins if $n \equiv 1, 2, 3, 4 \pmod{5}$.
   - Player II wins if $n \equiv 0 \pmod{5}$.

3. 1-2-3-4-5 Game:
   - Player I wins if $n \equiv 1, 2, 3, 4, 5 \pmod{6}$.
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4. 1-2-3-· · ·-k Game:
   - Player I wins if $n \equiv 1, 2, 3, \ldots, k \pmod{k+1}$.
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Patterns of Patterns!

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Player I wins if \( n \equiv 1, 2, 3 \pmod{4} \).

Player II wins if \( n \equiv 0 \pmod{4} \).

2. 1-2-3-4 Game:

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3. 1-2-3-4-5 Game:
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3. 1-2-3-4-5 Game:
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4. 1-2-3-⋯-k Game:
Patterns of Patterns!

Recall:

1. 1-2-3 Game:
   - Player I wins if $n \equiv 1, 2, 3 \pmod{4}$.
   - Player II wins if $n \equiv 0 \pmod{4}$

2. 1-2-3-4 Game:
   - Player I wins if $n \equiv 1, 2, 3, 4 \pmod{5}$.
   - Player II wins if $n \equiv 0 \pmod{5}$

3. 1-2-3-4-5 Game:
   - Player I wins if $n \equiv 1, 2, 3, 4, 5 \pmod{6}$.
   - Player II wins if $n \equiv 0 \pmod{6}$

4. 1-2-3-⋯-k Game:
Patterns of Patterns!

Recall:

1. 1-2-3 Game:
   ▶ Player I wins if \( n \equiv 1, 2, 3 \) (mod 4).
   ▶ Player II wins if \( n \equiv 0 \) (mod 4)

2. 1-2-3-4 Game:
   ▶ Player I wins if \( n \equiv 1, 2, 3, 4 \) (mod 5).
   ▶ Player II wins if \( n \equiv 0 \) (mod 5)

3. 1-2-3-4-5 Game:
   ▶ Player I wins if \( n \equiv 1, 2, 3, 4, 5 \) (mod 6).
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4. 1-2-3-⋯-\( k \) Game:
   ▶ Player I wins if
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1. 1-2-3 Game:
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   - Player I wins if $n \equiv 1, 2, 3, 4, 5 \pmod{6}$.
   - Player II wins if $n \equiv 0 \pmod{6}$

4. 1-2-3-⋯-k Game:
   - Player I wins if $n \equiv 1, 2, 3, \ldots, k \pmod{k+1}$. 
Patterns of Patterns!

Recall:

1. 1-2-3 Game:
   - Player I wins if \( n \equiv 1, 2, 3 \) (mod 4).
   - Player II wins if \( n \equiv 0 \) (mod 4)

2. 1-2-3-4 Game:
   - Player I wins if \( n \equiv 1, 2, 3, 4 \) (mod 5).
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3. 1-2-3-4-5 Game:
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   - Player II wins if \( n \equiv 0 \) (mod 6)

4. 1-2-3-\cdots-k Game:
   - Player I wins if \( n \equiv 1, 2, 3, \ldots, k \) (mod \( k + 1 \)).
   - Player II wins if
Patterns of Patterns!

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   ▶ Player II wins if $n \equiv 0 \pmod{k+1}$
More Patterns of Patterns!

Recall:

1. 1-3-4 Game:
   ▶ Player I wins if $n \equiv 1, 3, 4, 5, 6 \pmod{7}$.
   ▶ Player II wins if $n \equiv 0, 2 \pmod{7}$.

2. 1-4-5 Game:
   ▶ Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.
   ▶ Player II wins if $n \equiv 0, 2 \pmod{8}$.

3. 1-5-6 Game: I leave this to you.

4. General Pattern: I leave this to you.
More Patterns of Patterns!

Recall:

1. 1-3-4 Game:

   Player I wins if $n \equiv 1, 3, 4, 5, 6 \pmod{7}$.

   Player II wins if $n \equiv 0, 2 \pmod{7}$.

2. 1-4-5 Game:

   Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.

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More Patterns of Patterns!

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1. 1-3-4 Game:
   - Player I wins if \( n \equiv 1, 3, 4, 5, 6 \) (mod 7).

2. 1-4-5 Game:
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1. 1-3-4 Game:
   - ▶ Player I wins if $n \equiv 1, 3, 4, 5, 6 \pmod{7}$.
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2. 1-4-5 Game:
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3. 1-5-6 Game: I leave this to you

4. General Pattern: I leave this to you.
More Patterns of Patterns!

Recall:

1. 1-3-4 Game:
   ▶ Player I wins if $n \equiv 1, 3, 4, 5, 6 \pmod{7}$.
   ▶ Player II wins if $n \equiv 0, 2 \pmod{7}$

2. 1-4-5 Game:
   ▶ Player I wins if

3. 1-5-6 Game: I leave this to you

4. General Pattern: I leave this to you.
More Patterns of Patterns!

Recall:

1. 1-3-4 Game:
   - Player I wins if $n \equiv 1, 3, 4, 5, 6 \pmod{7}$.
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2. 1-4-5 Game:
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More Patterns of Patterns!

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More Patterns of Patterns!

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More Patterns of Patterns!

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1. 1-3-4 Game:
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   - Player II wins if $n \equiv 0, 2 \pmod{7}$

2. 1-4-5 Game:
   - Player I wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$.
   - Player II wins if $n \equiv 0, 2 \pmod{8}$

3. 1-5-6 Game: I leave this to you
More Patterns of Patterns!

Recall:

1. 1-3-4 Game:
   ▶ Player I wins if \( n \equiv 1, 3, 4, 5, 6 \) (mod 7).
   ▶ Player II wins if \( n \equiv 0, 2 \) (mod 7)

2. 1-4-5 Game:
   ▶ Player I wins if \( n \equiv 1, 3, 4, 5, 6, 7 \) (mod 8).
   ▶ Player II wins if \( n \equiv 0, 2 \) (mod 8)

3. 1-5-6 Game: I leave this to you

4. General Pattern: I leave this to you.
Player I wins if $n \equiv 1, 3, 5, 6, 7, 8, 9, 10 \pmod{11}$.

Player II wins if $n \equiv 0, 2, 4 \pmod{11}$.

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- Player I wins if
  \[ n \equiv 1, 3, 5, 6, 7, 8, 9, 10 \pmod{11} \]

- Player II wins if
  \[ n \equiv 0, 2, 4 \pmod{11} \]
Player I wins if $n \equiv 1, 3, 5, 6, 7, 8, 9, 10 \pmod{11}$. 
Player I wins if \( n \equiv 1, 3, 5, 6, 7, 8, 9, 10 \pmod{11} \).

Player II wins if
1-5-6

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- Player I wins if \( n \equiv 1, 3, 5, 6, 7, 8, 9, 10 \mod 11 \).
- Player II wins if \( n \equiv 0, 2, 4 \mod 11 \)
Player I wins if \( n \equiv 1, 3, 5, 6, 7, 8, 9, 10, 11 \pmod{11} \).

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- Player I wins if \( n \equiv 1, 3, 5, 6, 7, 8, 9, 10, 11 \pmod{11} \).
- Player II wins if \( n \equiv 0, 2, 4 \pmod{12} \).
Player I wins if \( n \equiv 1, 3, 5, 6, 7, 8, 9, 10, 11 \) (mod 11).
Player I wins if $n \equiv 1, 3, 5, 6, 7, 8, 9, 10, 11 \pmod{11}$.

Player II wins if $n \equiv 0, 2, 4 \pmod{12}$. 

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- Player I wins if \( n \equiv 1, 3, 5, 6, 7, 8, 9, 10, 11 \pmod{11} \).
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Patterns for $1-a-a+1$, $a$ odd

1-3-4
Patterns for $1-a-a+1$, $a$ odd

1-3-4

- Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
Patterns for $1-a-a+1$, $a$ odd

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Patterns for $1-a-a+1$, $a$ odd

1-3-4
- Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
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1-5-6

$1-a-a+1$ for $a$ odd
- Player I wins if $n \not\equiv 0, 2, \ldots, a-1 \pmod{2^a-1}$.
- Player II wins if $n \equiv 0, 2, \ldots, a-1 \pmod{2^a-1}$
Patterns for $1-a-a + 1$, $a$ odd

1-3-4
- Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
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1-5-6
- Player I wins if

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Patterns for $1-a-a+1$, $a$ odd

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   ▶ Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
   ▶ Player II wins if $n \equiv 0, 2 \pmod{7}$

1-5-6
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Patterns for $1-a-a+1$, $a$ odd

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Patterns for $1-a-a+1$, $a$ odd

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- Player I wins if $n \not\equiv 0, 2, 4 \pmod{11}$.
- Player II wins if $n \equiv 0, 2, 4 \pmod{11}$
Patterns for $1-a-a+1$, $a$ odd

1-3-4
- Player I wins if $n \not\equiv 0, 2$ (mod 7).
- Player II wins if $n \equiv 0, 2$ (mod 7)

1-5-6
- Player I wins if $n \not\equiv 0, 2, 4$ (mod 11).
- Player II wins if $n \equiv 0, 2, 4$ (mod 11)

1-7-8
Patterns for $1-a-a+1$, $a$ odd

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- Player I wins if $n \not\equiv 0, 2$ (mod 7).
- Player II wins if $n \equiv 0, 2$ (mod 7)

1-5-6
- Player I wins if $n \not\equiv 0, 2, 4$ (mod 11).
- Player II wins if $n \equiv 0, 2, 4$ (mod 11)

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Patterns for $1-a-a + 1$, $a$ odd

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- Player II wins if $n \equiv 0, 2 \pmod{7}$

1-5-6
- Player I wins if $n \not\equiv 0, 2, 4 \pmod{11}$.
- Player II wins if $n \equiv 0, 2, 4 \pmod{11}$

1-7-8
- Player I wins if $n \equiv 0, 2, 4, 6 \pmod{15}$. 
Patterns for $1-a-a+1$, $a$ odd

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- Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
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1-$a-a + 1$ for $a$ ODD
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1-$a$-$a+1$ for $a$ ODD
- Player I wins if
Patterns for $1-a-a+1$, $a$ odd

1-3-4

- Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
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1-5-6

- Player I wins if $n \not\equiv 0, 2, 4 \pmod{11}$.
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1-7-8

- Player I wins if $n \equiv 0, 2, 4, 6 \pmod{15}$.
- Player II wins if $n \equiv 0, 2, 4, 6 \pmod{15}$

1-\(a\)-\(a\) + 1 for $a$ ODD

- Player I wins if $n \not\equiv 0, 2, \ldots, a-1 \pmod{2a-1}$. 


Patterns for $1-a-a + 1$, $a$ odd

1-3-4

- Player I wins if $n \not\equiv 0, 2 \pmod{7}$.
- Player II wins if $n \equiv 0, 2 \pmod{7}$

1-5-6

- Player I wins if $n \not\equiv 0, 2, 4 \pmod{11}$.
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- Player I wins if $n \equiv 0, 2, 4, 6 \pmod{15}$.
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1-\(a\)-\(a\) + 1 for \(a\) ODD

- Player I wins if $n \not\equiv 0, 2, \ldots, a-1 \pmod{2a-1}$.
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1-$a$-$a+1$ for $a$ ODD
- Player I wins if $n \not\equiv 0, 2, \ldots, a-1 \pmod{2a-1}$.
- Player II wins if $n \equiv 0, 2, \ldots, a-1 \pmod{2a-1}$
Patterns for $1-a-a + 1$, $a$ even

1-2-3
Patterns for 1-a-a + 1, a even

1-2-3

- Player I wins if $n \not\equiv 0 \pmod{4}$. 
Patterns for $1-a-a+1$, $a$ even

1-2-3

- Player I wins if $n \not\equiv 0 \pmod{4}$.
- Player II wins if $n \equiv 0 \pmod{4}$.
Patterns for 1-\(a-a+1\), \(a\) even

1-2-3

- Player I wins if \(n \not\equiv 0 \pmod{4}\).
- Player II wins if \(n \equiv 0 \pmod{4}\)

1-4-5
Patterns for $1-a-a+1$, $a$ even

1-2-3
- Player I wins if $n \not\equiv 0 \pmod{4}$.
- Player II wins if $n \equiv 0 \pmod{4}$

1-4-5
- Player I wins if $n \not\equiv 0, 2 \pmod{8}$.
Patterns for $1-a-a+1$, $a$ even

1-2-3
  ▶ Player I wins if $n \not\equiv 0 \pmod{4}$.
  ▶ Player II wins if $n \equiv 0 \pmod{4}$

1-4-5
  ▶ Player I wins if $n \not\equiv 0, 2 \pmod{8}$.
Patterns for $1-a-a + 1, a$ even

1-2-3

- Player I wins if $n \not\equiv 0 \pmod{4}$.
- Player II wins if $n \equiv 0 \pmod{4}$

1-4-5

- Player I wins if $n \not\equiv 0, 2 \pmod{8}$.
- Player II wins if
Patterns for 1-\(a\)-\(a\) + 1, \(a\) even

1-2-3

- Player I wins if \(n \not\equiv 0 \pmod{4}\).
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1-4-5

- Player I wins if \(n \not\equiv 0, 2 \pmod{8}\).
- Player II wins if \(n \equiv 0, 2 \pmod{8}\)

1-2-\(a\) - 1 for \(a\) even

- Player I wins if \(n \not\equiv 0, 2, \ldots, a-2 \pmod{2a}\).
- Player II wins if \(n \equiv 0, 2, \ldots, a-2 \pmod{2a}\)
Patterns for $1-a-a+1$, $a$ even

1-2-3
- Player I wins if $n \not\equiv 0 \pmod{4}$.
- Player II wins if $n \equiv 0 \pmod{4}$

1-4-5
- Player I wins if $n \not\equiv 0, 2 \pmod{8}$.
- Player II wins if $n \equiv 0, 2 \pmod{8}$

1-6-7
Patterns for 1-\(a\)-a + 1, a even

1-2-3

- Player I wins if \(n \not\equiv 0 \pmod{4}\).
- Player II wins if \(n \equiv 0 \pmod{4}\)

1-4-5

- Player I wins if \(n \not\equiv 0, 2 \pmod{8}\).
- Player II wins if \(n \equiv 0, 2 \pmod{8}\)

1-6-7

- Player I wins if
Patterns for $1-a-a+1$, $a$ even

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- Player I wins if $n \equiv 0, 2, 4 \pmod{12}$. 
Patterns for 1-\(a\)-\(a\) + 1, \(a\) even

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1-a-a + 1 for a even
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- Player I wins if $n \not\equiv 0, 2, \ldots, a-2 \pmod{2a}$. 
Patterns for $1-a-a+1$, $a$ even

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