COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 15 (1) (1995) 85–104

ON ERDŐS–RADO NUMBERS

HANNO LEFMANN, and VOJTĚCH RÖDL*

Received Januray 28, 1993
Revised January 3, 1994

In this paper new proofs of the Canonical Ramsey Theorem, which originally has been proved by Erdős and Rado, are given. These yield improvements over the known bounds for the arising Erdős–Rado numbers $ER(k;l)$, where the numbers $ER(k;l)$ are defined as the least positive integer n such that for every partition of the k-element subsets of a totally ordered n-element set X into an arbitrary number of classes there exists an l-element subset Y of X, such that the set of k-element subsets of Y is partitioned canonically (in the sense of Erdős and Rado). In particular, it is shown that

$$2^{c_1 \cdot l^2} \leq ER(2;l) \leq 2^{c_2 \cdot l^2 \cdot \log l}$$

for every positive integer $l \geq 3$, where c_1, c_2 are positive constants. Moreover, new bounds, lower and upper, for the numbers $ER(k;l)$ for arbitrary positive integers k, l are given.

1. Introduction

In 1930 Ramsey proved his famous Theorem:

Theorem 1.1. [24] *Let k,l,t be positive integers. Then there exists a least positive integer $n = R_t(k;l)$ such that for every coloring $\Delta : [\{1,2,\ldots,n\}]^k \longrightarrow \{1,2,\ldots,t\}$ of the k-element subsets of $\{1,2,\ldots,n\}$ with t colors there exists a monochromatic l-element subset $X \subseteq \{1,2,\ldots,n\}$, i.e. $\Delta(S) = \Delta(T)$ for all $S,T \in [X]^k$.*

During the last few decades much interest has been drawn towards determining the growth rate of the Ramsey numbers $R_t(k;l)$. For $k=2$, it is known that $2^{c_1 \cdot l \cdot t} \leq R_t(2;l) \leq 2^{c_2 \cdot l \cdot t \cdot \log t}$ for positive integers $l \geq 3$ and $t \geq 2$, where c_1, c_2 are positive constants. For arbitrary positive integers k the numbers $R_t(k;l)$ grow like a tower function.

For positive integers k,l let $tow_k(l)$ denote the tower function of l with base 2 and height k, i.e.

$$tow_k(l) = 2^{2^{\cdot^{\cdot^{\cdot^{2^l}}}}}$$

with $(k-1)$ twos in the tower.

Erdős, Hajnal and Rado determined the following lower and upper bounds for the Ramsey numbers:

Mathematics Subject Classification (1991): 05 A 99

* Supported by NSF Grant DUS–9011850.

Theorem 1.2. [14], [10], [9] *Let k, t be positive integers with $k \geq 3$ and $t \geq 2$. Then there exist positive constants $c_{k,t}, c^*_{k,t}$ such that*

$$R_t(k;l) \leq tow_k(c^*_{k,t} \cdot l)$$

and

$$R_t(k;l) \geq tow_k(c_{k,t} \cdot l) \qquad \text{if } t \geq 4$$
$$R_3(k;l) \geq tow_{k-1}(c_{k,3} \cdot l^2 \cdot \log l)$$
$$R_2(k;l) \geq tow_{k-1}(c_{k,2} \cdot l^2),$$

provided $l \geq l_0(k)$.

Moreover, in [6] it has been shown that the lower bounds given in Theorem 1.2 still hold for small values of l, i.e. $t \geq 3$ and $k < l < l_0(k)$.

While in Ramsey's Theorem 1.1 the number t of colors is fixed, Erdős and Rado considered the case of arbitrary colorings of k-element subsets of a set X. They proved, that in this case, once $|X|$ is large enough, one can always find a subset of prescribed size, which is colored according to one of a few canonical patterns. In order to make this precise, we use the following notation:

Notation 1.1. *Let k be a positive integer and let $I \subseteq \{1,2,\ldots,k\}$ be a subset. Let $X = \{x_1, x_2, \ldots, x_k\}_<$ be a totally ordered set, i.e. $x_1 < x_2 < \ldots < x_k$.*

Then $X : I$ denotes the I-subset of X, i.e.

$$X : I = \{x_i \mid i \in I\}.$$

Theorem 1.3. [13] *Let k, l be positive integers with $l > k$. Then there exists a least positive integer $n = ER(k;l)$ such that for every coloring $\Delta: [\{1,2,\ldots,n\}]^k \longrightarrow \omega$ there exists a (possibly empty) subset $I \subseteq \{1,2,\ldots,k\}$ and there exists an l-element subset $X \subseteq \{1,2,\ldots,n\}$ such that for all k-element subsets $S,T \in [X]^k$ the following is valid:*

$$\Delta(S) = \Delta(T) \qquad \text{if and only if} \qquad S : I = T : I.$$

Thus for arbitrary colorings of the k-element subsets of a groundset X we still obtain on a subset of X some form of regularity given by 2^k canonical patterns. None of these patterns can be omitted without violating the statement in Theorem 1.3, as one can color just according to a missing pattern.

In this paper we will study the growth rate of the Erdős–Rado numbers $ER(k;l)$. For the simplest case of coloring singletons the value of $ER(1;l)$ is folklore, cf. also [18]:

Proposition 1.1. *Let l be a positive integer. Then*

$$ER(1;l) = (l-1)^2 + 1.$$

For colorings of k-element subsets, $k \geq 2$, the situation is less clear. The original proof of Erdős and Rado, as its simplification given by Rado in [23] uses Ramsey's Theorem for colorings of $2k$-element sets. Quite often, such a strategy turns out to be fruitful in proving canonical Ramsey-type theorems. On the other hand, this approach yields large upper bounds for the Erdős–Rado numbers $ER(k;l)$. In particular, the proof of Erdős and Rado as well as Rado's simplified version imply for the numbers $ER(k;l)$ upper bounds, which are tower functions of height $2k$.

Here we will give new proofs of the Erdős–Rado Theorem 1.3. Our approach yields the following lower and upper bounds for the numbers $ER(k;l)$:

Theorem 1.4. *Let k be a positive integer. Then there exist positive constants c_k, c_k^* such that for all positive integers l with $l \geq l_0(k)$ the following holds*

$$2^{c_2 \cdot l^2} \leq ER(2;l) \leq 2^{c_2^* \cdot l^2 \cdot \log l}$$

$$tow_k(c_k \cdot l^2) \leq ER(k;l) \leq tow_{k+1}(c_k^* \cdot \frac{l^{2k-1}}{\log l}) \qquad \text{if } k \geq 3.$$

2. Coloring pairs

In this section we consider arbitrary colorings of two-element sets, focussing our interest on the maximum size of canonically colored subsets. In particular, we investigate the growth of the Erdős–Rado numbers $ER(2;l)$.

Recall that for colorings of two-element subsets there are exactly four canonical coloring patterns: the monochromatic ($I = \emptyset$), the one-to-one ($I = \{1,2\}$), the minimum- ($I = \{1\}$) and the maximum-coloring ($I = \{2\}$). As usual, a one-to-one colored set $[X]^k$ or simply (by abuse of language) X will be called a totally multicolored set.

By using probabilistic methods, Galvin, cf. [16, p.30], obtained the following lower bound

$$ER(2;l) \geq \left(\frac{l}{\sqrt{2}} + o(1) \right)^l .$$

On the other hand, the upper bounds following from the original proof of Erdős and Rado [13] respective the proof of Rado [23] are three times exponential.

In [21] both bounds, lower and upper, have been improved:

Theorem 2.1. [21] *There exist positive constants $c, c' > 0$ such that for every positive integer $l \geq 3$ the following holds*

$$2^{c \cdot l^2} \leq ER(2;l) \leq 2^{2^{c' \cdot l^3}} .$$

Here we further improve the upper bound for $ER(2;l)$ given in Theorem 2.1. In particular, we give a new proof of Theorem 1.3 for colorings of two-element sets without using a higher dimensional version of Ramsey's Theorem.

Theorem 2.2. *There exist positive constants c_1, c_2 such that for all positive integers l with $l \geq 3$ the following holds*

$$2^{c_1 \cdot l^2} \leq ER(2;l) \leq 2^{c_2 \cdot l^2 \cdot \log l}.$$

Proof. The lower bound was proved already in [21]. For completeness we include the argument.

Claim 2.1.

$$ER(k;l) \geq R_{l-k}(k;l). \tag{1}$$

Proof. Put $n = R_{l-k}(k;l) - 1$ and let $\Delta: [\{1,2,\ldots,n\}]^k \longrightarrow \{1,2,\ldots,l-k\}$ be a coloring admitting no monochromatic l-element subset $X \subseteq \{1,2,\ldots,n\}$. Then, as there are only $l-k$ colors available, there is no l-element subset $X \subseteq \{1,2,\ldots,n\}$ and no subset $I \subseteq \{1,2,\ldots,k\}$, I nonempty, such that for all S, $T \in [X]^k$ it is valid that $\Delta(S) = \Delta(T)$ if and only if $S : I = T : I$ as each of these canonical colorings requires at least $l-k+1$ colors. By choice of the coloring Δ, no l-element subset $X \subseteq \{1,2,\ldots,n\}$ is monochromatic, which proves the Claim. ∎

As

$$2^{c_3 \cdot l \cdot t} \leq R_t(2;l) \leq 2^{c_4 \cdot l \cdot t \cdot \log t} \tag{2}$$

for all positive integers $l \geq 3$, where c_3, c_4 are positive constants (cf. [8], [17], [19]) we infer with (1) for $k=2$ that

$$ER(2;l) \geq R_{l-2}(2;l) \geq 2^{c_1 \cdot l^2}$$

for some positive constant c_1.

Next we prove the upper bound. Let l be a positive integer with $l \geq 3$. Let n be a positive integer with

$$n = \left\lceil \left(\frac{27 \cdot l^6}{16} \right)^{2(l-2)^2+1} \right\rceil$$

and let $\Delta: [\{1,2,\ldots,n\}]^2 \longrightarrow \omega$ be an arbitrary coloring.

Put with forsight $c = \frac{16}{27 \cdot l^6}$. We define nonempty subsets $V_L^{(i)}, V_R^{(i)}, M^{(i)} \subseteq \{1,2,\ldots,n\}$ for $i = 0,1,\ldots,s$, where s is some nonnegative integer with $s \leq 2 \cdot (l-2)^2 + 1$, as follows.

Set $V_L^{(0)} = V_R^{(0)} = \emptyset$ and put $M^{(0)} = \{1,2,\ldots,n\}$. Let j be a positive integer and assume that for each $i = 0,1,\ldots,j-1$ pairwise disjoint sets $V_L^{(i)}, V_R^{(i)}, M^{(i)} \subseteq \{1,2,\ldots,n\}$ with $V_L^{(i-1)} \subseteq V_L^{(i)}$, $V_R^{(i-1)} \subseteq V_R^{(i)}$, $|V_L^{(i)} \cup V_R^{(i)}| = i$ and $M^{(i)} \subset M^{(i-1)}$ as well as numbers $d_i \in \omega$, are already defined such that

(i) for $v \in V_L^{(i)} \setminus V_L^{(i-1)}$ and for all $w \in M^{(i)}$ it is valid that $v > w$ and $\Delta(\{w,v\}) = d_i$,

(ii) for $v \in V_R^{(i)} \setminus V_R^{(i-1)}$ and for all $w \in M^{(i)}$ it is valid that $v < w$ and $\Delta(\{v,w\}) = d_i$ and

(iii) $|M^{(i)}| \geq \lceil c \cdot |M^{(i-1)}| \rceil$.

Next we construct the sets $V_L^{(j)}, V_R^{(j)}, M^{(j)}$. Let $w \in M^{(j-1)}$ and $d \in \omega$. First define sets $X_L^d(w), X_R^d(w)$ (we omit the index j) as follows

$$X_L^d(w) = \{v \in M^{(j-1)} \mid \Delta(\{v,w\}) = d \text{ and } v < w\}$$
$$X_R^d(w) = \{z \in M^{(j-1)} \mid \Delta(\{w,z\}) = d \text{ and } z > w\}.$$

First assume that for some $w \in M^{(j-1)}$ and for some $d \in \omega$ we have $|X_L^d(w)| \geq \lceil c \cdot |M^{(j-1)}| \rceil$ or $|X_R^d(w)| \geq \lceil c \cdot |M^{(j-1)}| \rceil$. Then put $d_j = d$ and take any subset X with either $X \subseteq X_L^d(w)$ or $X \subseteq X_R^d(w)$, satisfying $|X| = \lceil c \cdot |M^{(j-1)}| \rceil$, and put $M^{(j)} = X$. If $X \subseteq X_L^d(w)$, put $V_L^{(j)} = V_L^{(j-1)} \cup \{w\}$ and $V_R^{(j)} = V_R^{(j-1)}$ and observe that $w > v$ for all $v \in V_L^{(j-1)}$. On the other hand, if $X \subseteq X_R^d(w)$, put $V_R^{(j)} = V_R^{(j-1)} \cup \{w\}$ and $V_L^{(j)} = V_L^{(j-1)}$, and we have that $w < v$ for all $v \in V_R^{(j-1)}$. Then we continue with step $(j+1)$.

If for all $w \in M^{(j-1)}$ and all $d \in \omega$ we have that $|X_L^d(w)| < \lceil c \cdot |M^{(j-1)}| \rceil$ and also that $|X_R^d(w)| < \lceil c \cdot |M^{(j-1)}| \rceil$, then the process stops.

Assume that this process continues until some step s, $s \leq 2 \cdot (l-2)^2 + 1$, where either $|V_L^{(s)}| = (l-2)^2 + 1$ or $|V_R^{(s)}| = (l-2)^2 + 1$.

Assume first that $|V_L^{(s)}| = (l-2)^2+1$, where $V_L^{(s)} = \{v_1, v_2, \ldots, v_{(l-2)^2+1}\}_<$. For $i = 1, 2, \ldots, (l-2)^2+1$ let $r(i)$ be the unique integer j such that v_i is an element of $V_L^{(j)}$ but not an element of $V_L^{(j-1)}$. The corresponding sequence $\left(d_{r(i)}\right)_{1 \leq i \leq (l-2)^2+1}$ contains by Proposition 1.1 a subsequence $\left(d_{r(i_j)}\right)_{1 \leq j \leq l-1}$ such that

(i) either $d_{r(i_j)} = d_{r(i_{j+1})}$ for $j = 1, 2, \ldots, l-2$

(ii) or $d_{r(i_j)} \neq d_{r(i_m)}$ for all integers j, m with $1 \leq j < m \leq l-1$.

Then for $r = r(i_{l-1}) = \max_{1 \leq j \leq l-1} r(i_j)$ the l-element set

$$\left\{v_{r(i_1)}, v_{r(i_2)}, \ldots, v_{r(i_{l-1})}, v\right\},$$

where $v \in M^{(r)}$ is arbitrary, is in case (i) monochromatic and in case (ii) it is maximum-colored with respect to Δ.

The arguments for the second case, $|V_R^{(s)}| = (l-2)^2+1$, are similar to those used above. We obtain an l-element subset $X \subseteq V_R^{(s)}$, which is either monochromatic or minimum-colored.

Now assume that the process stops at some step s, $s < 2 \cdot (l-2)^2 + 1$. The set $Y = M^{(s)}$ satisfies $|Y| = m \geq \lceil \frac{1}{c} \rceil = \frac{27}{16} \cdot l^6$. For $y \in Y$ and $d \in \omega$ let $\deg_y^{(d)} = |\{w \in Y \mid \Delta(\{w, y\}) = d\}|$. As the process stops, we have

$$\deg_y^{(d)} < 2c \cdot m \tag{3}$$

and, clearly,

$$\sum_{d \in \omega} \deg_y^{(d)} = m - 1 \tag{4}$$

for all $y \in Y$ and all $d \in \omega$.

A three-element subset $T \in [Y]^3$ is called a *bad triple*, if the set $[T]^2$ is not totally multicolored with respect to Δ. Let b denote the number of bad triples in $[Y]^3$. Then

$$b \leq \sum_{y \in Y} \sum_{d \in \omega} \binom{\deg_y^{(d)}}{2}.$$

The sum $\sum_{d \in \omega} \binom{\deg_y^{(d)}}{2}$ subject to (3) and (4) is maximal if all summands are as large as possible, hence we infer that

$$\sum_{d \in \omega} \binom{\deg_y^{(d)}}{2} < \frac{m}{2cm} \cdot \binom{2cm}{2}$$

and therefore

$$b < \sum_{y \in Y} \frac{1}{2c} \cdot \binom{2cm}{2} = \frac{m}{2c} \cdot \binom{2cm}{2}. \tag{5}$$

Let Z^* be an $\lceil \frac{3}{4} \cdot l^3 \rceil$-element subset of Y, picked uniformly at random among all $\lceil \frac{3}{4} \cdot l^3 \rceil$-element subsets of Y. Let E denote the expected number of bad triples in Z^*. Using (5) and that $c = \frac{16}{27 \cdot l^6}$ we obtain that

$$E = \frac{b \cdot \binom{m-3}{\lceil \frac{3}{4} \cdot l^3 \rceil - 3}}{\binom{m}{\lceil \frac{3}{4} \cdot l^3 \rceil}} < \frac{b}{m^3} \cdot \left(\frac{3}{4} l^3\right)^3 < \frac{27}{64} \cdot c \cdot l^9 = \frac{l^3}{4}$$

and hence there exists a subset $Z^* \in [Y]^{\lceil \frac{3}{4} \cdot l^3 \rceil}$ having at most $\frac{1}{4} \cdot l^3$ bad triples. Delete one vertex from each bad triple. Then the remaining subset $Z \subseteq Z^*$ contains no bad triple anymore and has size $|Z| \geq \frac{1}{2} \cdot l^3$. Thus for every three-element subset $T \in [Z]^3$, the set $[T]^2$ is totally multicolored. In terms of graphs, every color class in $[Z]^2$ is a matching.

We use the following result of Babai:

Lemma 2.1. [4] *Let m be a positive integer. Let $\Delta: [\{1,2,\ldots,m\}]^2 \longrightarrow \omega$ be a coloring, where every color class is a matching.*

Then there exists a totally multicolored subset $X \subseteq \{1,2,\ldots,m\}$ with

$$|X| \geq (2 \cdot m)^{\frac{1}{3}}. \tag{6}$$

We include the proof of this lemma for completeness.

Proof. Let $M = \{1,2,\ldots,m\}$ and let $\Delta: [M]^2 \longrightarrow \omega$ be a coloring, where every color class is a matching.

The proof uses a Greedy type argument. Let X be a maximal totally multicolored subset of M, and let $D = \{\Delta(\{x,x^*\}) \mid \{x,x^*\} \in [X]^2\}$ be the set of the occurring colors. By maximality of X, for every element $y \in M \setminus X$ there exists an element $x \in X$ such that $\Delta(\{x,y\}) \in D$. Moreover, as different two-element subsets of X, which are colored the same, cannot intersect nontrivially, we have the following inequality

$$m - |X| \leq |X| \cdot \binom{|X|}{2}.$$

Hence, as $l \cdot \binom{l}{2} \leq \frac{l^3}{2} - l$ for positive integers $l \geq 2$, it follows that

$$|X| \geq (2 \cdot m)^{\frac{1}{3}}. \qquad \blacksquare$$

Now, the restriction of the coloring Δ to the set $[Z]^2$ satisfies the assumptions of Lemma 2.1, and by (6) there exists a totally multicolored subset $X \subseteq Z$, with $|X| \geq (2 \cdot |Z|)^{\frac{1}{3}} \geq l$. This finishes the proof of Theorem 2.2. $\blacksquare$

The lower bound (6) was improved in [3] to

$$|X| \geq c \cdot m^{\frac{1}{3}} \cdot (\log m)^{\frac{1}{3}}, \tag{7}$$

where c is a positive constant. We remark that by using in the proof of Theorem 2.2 the inequality (7) instead of (6) we obtain a slight improvement on the upper bound of $ER(2;l)$, namely from $\left(c_1 \cdot l^6\right)^{2(l-2)^2+1}$ to

$$\left(\frac{c_2 \cdot l^6}{(\log l)^2}\right)^{2(l-2)^2+1}.$$

We have seen that

$$2^{c_1 \cdot l^2} \leq ER(2;l) \leq 2^{c_2 \cdot l^2 \cdot \log l}$$

for positive constants c_1, $c_2 > 0$. Moreover, the corresponding Ramsey numbers $R_{l-2}(2;l)$, which we used in the lower bound satisfy by (2) that

$$2^{c_3 \cdot l^2} \leq R_{l-2}(2;l) \leq 2^{c_4 \cdot l^2 \cdot \log l} \tag{8}$$

for some positive constants $c_3, c_4 > 0$.

By (2), at least $ER(2;l) \leq R_{c \cdot l \cdot \log l}(2;l)$ for some positive constant $c > 0$. Notice, that a totally multicolored l-element subset requires at least $\binom{l}{2}$ colors. Possibly, only $c \cdot l$ colors, which occur 'often', are crucial for the occurence of canonically (not totally multicolored) colored subsets and cause the growth of the Erdős–Rado numbers. Moreover, possibly the inequality $ER(2;l) \leq R_{c \cdot l}(2;l)$ holds. If true, decreasing the gap between lower and upper bound in (8) (also asked for in [17]) could give more insight here.

By (1) we have $ER(2;l) \geq R_{l-2}(2;l)$. This lower bound can be a little improved:

Fact 2.1. *Let l be a positive integer with $l \geq 4$. Then*

$$ER(2;l) \geq R_{l-2}(2;l) + l. \tag{9}$$

Proof. Let $n=R_{l-2}(2;l)-1$. Let $\Delta:[\{1,2,\ldots,n\}]^2 \longrightarrow \{1,2,\ldots,l-2\}$ be a coloring admitting no monochromatic l-element subset, where Δ is such that among all choices of such colorings color 1 is used as often as possible. We claim that $\{1,2,\ldots,n\}$ contains an l-element *ex-subset* Y, by which we mean an l-element subset, such that all, with the exception of one, two-element subsets of Y are colored the same.

Namely, each two-element subset $\{x,y\}$ of $\{1,2,\ldots,n\}$, not colored in color 1, can be extended to an l-element ex-set Y, where $[Y]^2 \setminus \{x,y\}$ is monochromatic in color 1, as otherwise the set $\{x,y\}$ can be recolored by color 1, increasing the size of color class 1.

Assume w.l.o.g. that $Y=\{1,2,\ldots,l\}$ is an ex-set with, say, $\{1,l\}$ colored differently from the other two-element subsets. Let $\{n+1,n+2,\ldots,n+l\}$ be a copy of Y. We extend the coloring Δ as follows. For $i \neq j$ color all two-element sets $\{j,n+i\}$ respective $\{n+i,n+j\}$ by the color $\Delta(\{i,j\})$ and color all sets $\{i,n+i\}$ by a new color. Clearly, the resulting coloring yields no canonically colored l-element subset. ∎

A similar argument was (in slightly more developed form) applied in [21].

As for the exact values of the Ramsey numbers $R_t(k;l)$, not much is known about the precise values of the Erdős–Rado numbers $ER(2;l)$ for small positive integers $l \geq 3$. We only know the exact value of $ER(2;3)$, namely:

Fact 2.2. $ER(2;3)=4$.

Proof. The coloring $\Delta:[\{1,2,3\}]^2 \longrightarrow \{1,2\}$ with $\Delta(\{1,2\})=\Delta(\{2,3\}) \neq \Delta(\{1,3\})$ gives the lower bound $ER(2;3) \geq 4$. To see the upper bound, let $\Delta:[\{1,2,3,4\}]^2 \longrightarrow \omega$ be a coloring. Suppose that for some positive integers a,b,c with $a<b<c \leq 4$ we have that $\Delta(\{a,b\}) \neq \Delta(\{b,c\})$. Then, in any case, the set $\{a,b,c\}$ is either minimum- or maximum- or totally multicolored. Therefore, we can assume that

$$\Delta(\{a,b\}) = \Delta(\{b,c\}) \qquad \text{for all } 1 \leq a < b < c \leq 4, \tag{10}$$

in particular, $\Delta(\{1,2\}) = \Delta(\{2,3\}) = \Delta(\{3,4\})$. Moreover, by (10) we have $\Delta(\{1,2\})=\Delta(\{2,4\})$, hence the set $\{2,3,4\}$ is monochromatic. ∎

With (9) we obtain the lower bound $ER(2;4) \geq 22$. A large upper bound (i.e. $2^{72} \cdot 3^{27}$) follows by Theorem 2.2.

3. Matchings and Stars

In the proof of Theorem 2.2 we used Lemma 2.1, a so called anti-Ramsey theorem. While in quantitative Ramsey-type theorems usually the size of a monochromatic subset is estimated, anti-Ramsey theorems are dealing with the size of a largest totally multicolored subset, compare also [2], [12], [15], [22], [25].

As already mentioned above, the lower bound in Lemma 2.1 was improved in [3]. Moreover, Babai also obtained an upper bound by using certain random colorings:

Theorem 3.1. [4], [3] *Let n be a positive integer. Let $\Delta\colon[\{1,2,\dots,n\}]^2 \longrightarrow \omega$ be a coloring, where every color class is a matching. Then there exists a totally multicolored subset $X \subseteq \{1,2,\dots,n\}$ with*

$$|X| \geq c\cdot(n\cdot\log n)^{\frac{1}{3}},$$

where c is a positive constant.

Moreover, for $n \geq n_0$, there exists a coloring $\Delta\colon[\{1,2,\dots,n\}]^2 \longrightarrow \omega$, where each color class is a matching, such that every totally multicolored subset $X \subseteq \{1,2,\dots,n\}$ satisfies

$$|X| \leq 8\cdot(n\cdot\log n)^{\frac{1}{3}}.$$

We remark, that Theorem 3.1 is related to a problem about the size of Sidon sets in Abelian groups, cf. [4].

While Babai considered colorings of two-elements sets, where every color class is a matching, we consider here a dual problem, namely colorings of the two-element subsets of a set N, where every color class is a star.

Definition 3.1. Let N be a set. A subset $S\subseteq[N]^2$ is a star if and only if $|\bigcap_{s\in S} s|\geq 1$.

Let $\Delta\colon[\{1,2,\dots,n\}]^2 \longrightarrow \omega$ be a coloring, where every color class is a star, and let l be a positive integer with $l\geq 3$. Clearly, no l-element subset of $\{1,2,\dots,n\}$ can be monochromatic. On the other hand, it follows from the Erdős–Rado Theorem 1.3 that for $n\geq ER(2;l)$ the following is valid:
(*) for every coloring $\Delta\colon[\{1,2,\dots,n\}]^2 \longrightarrow \omega$ there exists an l-element subset $X \subseteq \{1,2,\dots,n\}$, which is either minimum-colored or maximum-colored or totally multicolored.

Let $S(2;l)$ denote the least positive integer n such that (*) is true. By the observation above we have $S(2;l) \leq ER(2;l)$. With Theorem 2.2 this gives an upper bound of the order $2^{c\cdot l^2\cdot\log l}$ for some positive constant c. Indeed, one can get a much smaller upper bound for $S(2;l)$ of the order $2^{c^*\cdot l\cdot\log l}$ for some positive constant c^*, as the following result shows.

Theorem 3.2. *There exist positive constants c_1, c_2 such that for all positive integers l with $l\geq 3$ the following holds*

$$2^{c_1\cdot l\cdot\log l} \leq S(2;l) \leq 2^{c_2\cdot l\cdot\log l}.$$

In our arguments for proving the lower bound of $S(2;l)$ we use the Erdős Lovász Local Lemma. Let $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_n$ be events in a probability space Ω. A graph G on $\{1,2,\dots,n\}$ is a dependency graph for $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_n$ if for every $i=1,2,\dots,n$ the event $\mathcal{A}_i$ is mutually independent of $\{\mathcal{A}_j \mid \{i,j\}\notin E(G)\}$, cf. [26].

Lemma 3.1. [11] *Let $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_n$ be events in a probability space Ω with Prob $(\mathcal{A}_i) \leq p < 1$ for $i=1,2,\dots,n$ and with a dependency graph of maximum degree d.*

If

$$e\cdot p\cdot(d+1)\leq 1,$$

then

$$Prob\ \left(\wedge_{i=1}^{n}\bar{\mathscr{A}}_i\right) > 0.$$

Proof. Fix a positive integer $l \geq 3$. Let n be a positive integer with

$$n \leq \frac{1}{e} \cdot (l-2)^{\frac{l+1}{2}} \cdot \left(\frac{1}{e \cdot \binom{l}{2}}\right)^{\frac{1}{l-2}}. \tag{11}$$

Put $N = \{1,2,\dots,n\}$. For each element $v \in N$ choose an $(l-2)$-element subset $C_v \subset \omega$ of colors, where $C_v \cap C_{v^*} = \emptyset$ for each two distinct elements $v, v^* \in N$.

Define a random coloring $\Delta: [N]^2 \longrightarrow \omega$ as follows. For each pair $\{v^*, v\}$ with $v^*, v \in N$ and $v^* < v$ pick at random a color from C_v with probability $p = \frac{1}{l-2}$ and color the set $\{v^*, v\}$ by this color. Then for this coloring every color class is a star. Observe that due to the structure of the coloring classes every l-element subset X of N is neither monochromatic nor minimum-colored nor totally multicolored. For an l-element subset $L \in [N]^l$ let $\mathscr{A}_L$ denote the event that $[L]^2$ is maximum-colored. Thus if we prove that Prob $\left(\bigwedge_{L \in [N]^l} \bar{\mathscr{A}}_L\right) > 0$, we can infer that there exists a coloring $\Delta: [N]^2 \longrightarrow \omega$ such that every l-element subset $L \subseteq N$ is neither monochromatic, nor minimum-colored nor maximum-colored nor totally multicolored.

Now

$$\begin{aligned}\text{Prob } (\mathscr{A}_L) &= \prod_{i=1}^{l-2} p^i \\ &= \left(\frac{1}{l-2}\right)^{\binom{l-1}{2}}.\end{aligned}$$

Moreover, for each set $L \in [N]^l$ the event $\mathscr{A}_L$ is independent of all events $\mathscr{A}_{L^*}$, $L^* \in [N]^l$, if $|L \cap L^*| \leq 1$. Define a dependency graph G for the events $\mathscr{A}_L$, $L \in [N]^l$, with vertex set $[N]^l$ and edges $\{L, L^*\}$ if $|L \cap L^*| \geq 2$. The maximum degree d of this dependency graph G satisfies

$$d \leq \binom{l}{2} \cdot \binom{n-2}{l-2} - 1.$$

For n as in (11), we obtain that

$$\begin{aligned}\text{Prob } (\mathscr{A}_L) \cdot (d+1) \cdot e &< \left(\frac{1}{l-2}\right)^{\binom{l-1}{2}} \cdot \binom{l}{2} \cdot \binom{n-2}{l-2} \cdot e \\ &< \left(\frac{1}{l-2}\right)^{\binom{l-1}{2}} \binom{l}{2} \cdot \left(\frac{n \cdot e}{l-2}\right)^{l-2} \cdot e \\ &\leq 1,\end{aligned}$$

and hence, the assumptions of the Local Lemma 3.1 are satisfied.

The proof of the upper bound is similar to the proof of Theorem 2.2. We only sketch the arguments. Let l be a positive integer with $l \geq 3$ and let n be a positive integer with

$$n = \left\lceil \left(\frac{27 \cdot l^2}{4} \right)^{2l-3} \right\rceil .$$

Let $\Delta: [\{1,2,\ldots,n\}]^2 \longrightarrow \omega$ be a coloring, where every color class is a star. Put $c = \frac{4}{27 \cdot l^2}$. As in the proof of Theorem 2.2 define nonempty subsets $V_L^{(i)}, V_R^{(i)}, M^{(i)} \subseteq \{1,2,\ldots,n\}$ for $i = 0,1,\ldots,s$, where s is some nonnegative integer s with $s \leq 2l-3$ as long as possible. Having done this, we distinguish similarily as above between the following situations.

First assume that this process continues until step s, $s \leq 2l-3$, where either $|V_L^{(s)}| = l-1$ or $|V_R^{(r)}| = l-1$. If $|V_L^{(s)}| = l-1$ holds, then we obtain an l-element set, which is maximum-colored with respect to Δ. If $|V_R^{(s)}| = l-1$, then we get an l-element set X, which is minimum-colored with respect to Δ.

Finally, if the process stops at some step s, $s < 2l-3$, then the set $Y = M^{(s)}$ satisfies $|Y| = m \geq \lceil \frac{1}{c} \rceil \geq \frac{27}{4} \cdot l^2$. As in the proof of Theorem 2.2, by estimating the number of *bad triples* in a random subset $Z^* \subseteq Y$ with $|Z^*| = \lceil \frac{3}{2} l \rceil$, and then deleting one point from each bad triple, we obtain a totally multicolored subset $Z \subseteq Z^*$ with $|Z| \geq l$, as by assumption every color class is a star. ∎

4. Coloring k-sets

For colorings of three-element subsets the following bounds for the numbers $ER(3;l)$ are known.

Theorem 4.1. [20] *There exist positive constants c_1, c_2 such that for all positive integers l with $l \geq l_0$ the following holds*

$$2^{2^{c_1 \cdot l^2}} \leq ER(3;l) \leq 2^{2^{2^{c_2 \cdot l^5}}} .$$

In this section we will generalize and improve this result, by considering colorings of k-element sets for arbitrary fixed values of k:

Theorem 4.2. *Let $k \geq 3$ be a positive integer. Then there exist positive constants c_k, c_k^* such that for all positive integers l with $l \geq l_0(k)$ it is valid that*

$$tow_k(c_k \cdot l^2) \leq ER(k;l) \leq tow_{k+1}\left(c_k^* \cdot \frac{l^{2k-1}}{\log l} \right).$$

In analogy with the case for coloring pairs and supported by a result of Baumgartner [5] in the infinite case the lower bound for $k \geq 3$ should give the correct order. The approach in [5], however, does not seem to be applicable in the finite case.

Proof. By Claim 2.1 we have that $ER(k;l) \geq R_{l-k}(k;l)$. Using the lower bound $R_t(2;l) \geq 2^{c \cdot l \cdot t}$ from (2), where c is a positive constant and $l,t \geq 2$ are positive integers, and applying the same techniques as used to prove Theorem 1.2 it follows that $R_t(k;l) \geq tow_k(c'_k \cdot l \cdot t)$ for $t \geq 4$, hence, $ER(k;l) \geq R_{l-k}(k;l) \geq tow_k(c_k \cdot l^2)$ for $l \geq l_0(k)$ for some positive constant c_k.

In the following we will prove the upper bound. Let k,l be positive integers with $l \geq l_0(k)$. Set

$$n = R_{(k+1)^{k+1}}\left(k+1; C_k \cdot \frac{l^{2k-1}}{\log l}\right),$$

where C_k is a positive constant, which is large enough such that the following computations are valid. We will show that $ER(k;l) \leq n$. By Theorem 1.2 this implies that $ER(k;l) \leq tow_{k+1}(c_k^* \cdot l^{2k-1}/\log l)$ for some positive constant c_k^*.

Let $\Delta : [\{1,2,\ldots,n\}]^k \longrightarrow \omega$ be an arbitrary coloring. This coloring induces another coloring $\hat{\Delta} : [\{1,2,\ldots,n\}]^{k+1} \longrightarrow \omega$ as follows: for $(k+1)$-element subsets $Z = \{z_1, z_2, \ldots, z_{k+1}\}_<$ of $\{1,2,\ldots,n\}$, let

$$\hat{\Delta}(Z) = \begin{cases} I & \text{if } \Delta(S) = \Delta(T) \text{ if and only if } S:I = T:I \text{ for all } S,T \in [Z]^k \\ P & \text{if the above is not valid, and } P \text{ is the set of all pairs} \\ & \{(i,j) \mid 1 \leq i < j \leq k+1\} \text{ such that } \Delta(Z \setminus \{z_i\}) = \Delta(Z \setminus \{z_j\}). \end{cases}$$

Essentially, $\hat{\Delta}$ colors the $(k+1)$-element sets according to the equivalence relation on its k-element subsets induced by the coloring Δ. Let us point out that all cases when $\hat{\Delta}(Z) = I$ can be described by some equivalence relation P on pairs. It is however convenient for our purpose to consider "I-cases" separately.

Let us also note that for $I \subseteq \{1,2,\ldots,k\}$ the corresponding *P-pattern* can be described as follows:

Proposition 4.1. *Let $Z = \{z_1, z_2, \ldots, z_{k+1}\}_<$ and let $\Delta : [Z]^k \longrightarrow \omega$ be a coloring. Then the following statements are equivalent:*

(a) *There exists $I = \{i_1, i_2, \ldots, i_p\}_<$ with $I \subseteq \{1,2,\ldots,k\}$ such that $\Delta(S) = \Delta(T)$ if and only if $S:I = T:I$ for all S, $T \in [Z]^k$.*

(b) *$P = \cup_{l=1}^{p+1} P_l$, where $P_l = [\{i_{l-1}+1, i_{l-1}+2, \ldots, i_l\}]^2$, and $i_0 = 0$ and $i_{p+1} = k+1$. (In other words, $\Delta(Z \setminus \{z_i\}) = \Delta(Z \setminus \{z_j\})$, whenever $(i,j) \in P_l$ for some $l \in \{1,2,\ldots,p+1\}$.)*

Proof. This follows from the fact that for $S,T \in [Z]^k$, we have $S:I = T:I$, whenever $S = Z \setminus \{z_i\}$, $T = Z \setminus \{z_j\}$ and $(i,j) \in P_l$. ∎

By choice of n there exists a subset $X \subseteq \{1,2,\ldots,n\}$, $X = \{x_1, x_2, \ldots, x_m\}_<$, with

$$|X| = m \geq C_k \cdot \frac{l^{2k-1}}{\log l},$$

which is monochromatic with respect to the coloring $\hat{\Delta}$ in some color C. According to the value of C we distinguish two cases.

Case 1: $C = I$ Assume first that $I = \emptyset$. Then for every $(k+1)$-element subset Y of X the set $[Y]^k$ is monochromatic with respect to the coloring Δ. We claim that the whole set X is monochromatic with respect to Δ. Namely, suppose not, and let $K = \{x_1, x_2, \ldots, x_k\}$. By assumption there exist subsets S, S' with $S \subseteq K$, $S' \subseteq X \setminus K$ and $|S \cup S'| = k$ such that

$$\Delta(K) \neq \Delta(S \cup S').$$

Assume that among all such choices of subsets $S \cup S'$ the set S has largest possible cardinality. But then by maximality of $|S|$, for any element $s \in K \setminus S$, the $(k+1)$-element set $S \cup \{s\} \cup S'$ is not monochromatic, as $\Delta(K) = \Delta(S \cup \{s\} \cup S' \setminus \{s'\}) \neq \Delta(S \cup S')$ for every $s' \in S'$. This contradicts our assumption, hence $[X]^k$ is monochromatic.

We assume in the following that $I \neq \emptyset$. Let $X^* = \{x_{i \cdot k} \mid 1 \leq i \leq m^*\}$, where $m^* = \lfloor \frac{m-k+1}{k} \rfloor \}$. Define a coloring $\Delta^* : [X^*]^{|I|} \longrightarrow \omega$ by

$$\Delta^*(S) = \Delta(S \cup S') \qquad \text{for some } S' \subseteq X \setminus X^*,\ |S'| = k - |I| \text{ and } (S \cup S') : I = S.$$

Claim 4.1. *The coloring Δ^* is well defined.*

Proof. Suppose for contradiction that Δ^* is not well defined. Then for some subset $S \in [X^*]^{|I|}$ there exist two sets $S_1, S_2 \subseteq X \setminus X^*$ which satisfy

$$|S \cup S_1| = |S \cup S_2| = k$$
$$(S \cup S_1) : I = (S \cup S_2) : I = S$$
$$\Delta(S \cup S_1) \neq \Delta(S \cup S_2).$$

Let

$$S \cup S_1 = \{y_1, y_2, \ldots, y_k\}_<$$
$$S \cup S_2 = \{z_1, z_2, \ldots, z_k\}_<$$

and let g be the first position where $S \cup S_1$ and $S \cup S_2$ differ, i.e. $y_i = z_i$ for $i = 1, 2, \ldots, g-1$ and $y_g \neq z_g$. Among all such possible choices of pairs of different sets S_1 and S_2, let S_1 and S_2 be chosen such that g is maximal. Suppose w.l.o.g that $y_g < z_g$. As $g \notin I$, we infer by considering the set $T = S \cup S_2 \cup \{y_g\}$ that

$$\big(T \setminus \{y_g\}\big) : I = \big(T \setminus \{z_g\}\big) : I,$$

hence, by definition of the coloring $\hat{\Delta}$, we obtain that

$$\Delta(T \setminus \{y_g\}) = \Delta(T \setminus \{z_g\}).$$

This implies

$$\Delta(S \cup S_1) \neq \Delta(S \cup \{y_g\} \cup (S_2 \setminus \{z_g\})),$$

but the sets $S \cup S_1$ and $S \cup \{y_g\} \cup (S_2 \setminus \{z_g\}))$ do not differ on the first g positions, which contradicts the maximality of g. ∎

Moreover, the coloring Δ^* has the following property:

Claim 4.2. *Let $Y\in[X^*]^{|I|+1}$ be a set with $Y=\{y_1,y_2,\dots,y_{|I|+1}\}_<$. Then for every positive integer $j\le|I|$ it is valid that*

$$\Delta^*(Y\setminus\{y_j\})\neq\Delta^*(Y\setminus\{y_{j+1}\}).$$

Proof. Let j be a positive integer with $j\le|I|$. Let $S'\in[X]^{k-|I|}$ be a subset, which satisfies $S'\cap Y=\emptyset$, $(Y\setminus\{y_j\}\cup S'):I=Y\setminus\{y_j\}$, $(Y\setminus\{y_{j+1}\}\cup S'):I=Y\setminus\{y_{j+1}\}$ and $S'\cap\{y_j,y_j+1,\dots,y_{j+1}\}=\emptyset$. As $\hat{\Delta}(Y\cup S')=I$, it follows from $(Y\setminus\{y_j\}\cup S'):I\neq(Y\setminus\{y_{j+1}\}\cup S'):I$ that $\Delta(Y\setminus\{y_j\}\cup S')\neq\Delta(Y\setminus\{y_{j+1}\}\cup S')$. By definition of Δ^*, this implies $\Delta^*(Y\setminus\{y_j\})\neq\Delta^*(Y\setminus\{y_{j+1}\})$. ∎

Lemma 4.1. *Let g be a positive integer and let X^* be a totally ordered set with $|X^*|=m$, $m\ge m_0(g)$. Let $\Delta:[X^*]^g\longrightarrow\omega$ be a coloring with the following property: for every $(g+1)$-element subset $Y=\{y_1,y_2,\dots,y_{g+1}\}_<\in[X^*]^{g+1}$ and each positive integer j, $j\le g$, it is valid that*

$$\Delta(Y\setminus\{y_j\})\neq\Delta(Y\setminus\{y_{j+1}\}).\tag{12}$$

Then there exists a totally multicolored subset $Z\subseteq X^*$, with

$$|Z|\ge c(g)\cdot(m\cdot\log m)^{\frac{1}{2g-1}}$$

for some positive constant $c(g)$.

For the proof of Lemma 4.1 we will use the concept of uncrowded hypergraphs, cf. [1], [3]. Let $\mathcal{H}=(V,\mathcal{E})$ be a hypergraph with vertex set V and edge set $\mathcal{E}$. For a vertex $v\in V$, let $d_{\mathcal{H}}(v)=|\{E\in\mathcal{E}\mid v\in E\}|$ denote the degree of v. Let $D_{\mathcal{H}}=\max\{d_{\mathcal{H}}(v)\mid v\in V\}$ be the maximum degree of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is called k-uniform if $|E|=k$ for each edge $E\in\mathcal{E}$. A 2-cycle in $\mathcal{H}$ is given by two distinct edges from $\mathcal{E}$, which intersect in at least two vertices. The independence number $\alpha(\mathcal{H})$ is the maximum cardinality of a subset of V, which contains no edges from $\mathcal{E}$.

In the proof of Lemma 4.1 we will use the following theorem, which is a generalization of a deep result of Ajtai, Komlós, Pintz, Spencer and Szemerédi [1]:

Theorem 4.3. [7] *Let $\mathcal{H}=(V,\mathcal{E})$ be a k-uniform hypergraph with $|V|=n$ and maximum degree $D_{\mathcal{H}}\le t^{k-1}$. If*
(i) *$\mathcal{H}$ contains no 2-cycles, and*
(ii) *$t\gg k$,*
then

$$\alpha(\mathcal{H})\ge c_k\cdot\frac{n}{t}\cdot(\log t)^{\frac{1}{k-1}},\tag{13}$$

where c_k is a positive constant.

Next we will give the proof of Lemma 4.1.

Proof. Let $X^*=\{x_1,x_2,\dots,x_m\}_<$ be given with $x_1<x_2<\dots<x_m$. Let $\Delta:[X^*]^g\longrightarrow\omega$ be a coloring, which satisfies (12) for every subset $Y\in[X^*]^{g+1}$ and for every positive integer $j\le g$. Let $T=\{i\in\omega\mid\Delta^{-1}(i)\neq\emptyset\}$ be the set of all

occuring colors. Fix a color $t \in T$. Let $S \in [X^*]^{g-1}$ with $S = \{s_1, s_2, \ldots, s_{g-1}\}_<$ be arbitrary. Let j be a positive integer with $j \le g$. By (12) there do not exist two distinct elements $x, x' \in X^*$ with $s_{j-1} < x, x' < s_j$ such that

$$\Delta(S \cup \{x\}) = \Delta(S \cup \{x'\}) = t.$$

Hence, for every color $t \in T$ and for every $(g-1)$-element subset $S \in [X^*]^{g-1}$ there exist at most g elements $x_i' \in X^* \setminus S$, $i = 1, 2, \ldots, f$, with

$$\Delta(S \cup \{x_i'\}) = t \qquad \text{for } i = 1, 2, \ldots, f \text{ with } f \le g \tag{14}$$

and every subset $G \in [X^*]^g$ with $\Delta(G) = t$ arises from some set $S \in [X^*]^{g-1}$ in this way.

Let j be a nonnegative integer and let $\{G, G^*\}$ be a two-element subset of $[X^*]^g$. The (unordered) pair $\{G, G^*\}$ is called a *j-pair* if $|G \cap G^*| = j$, and it is called a *bad pair* if $\Delta(G) = \Delta(G^*)$.

For nonnegative integers $j = 0, 1, \ldots, g-1$ fix a j-element subset $J \in [X^*]^j$. For nonnegative integers $t \in T$ put $r_t(J) = |\{G \in [X^*]^g \mid \Delta(G) = t \text{ and } J \subset G\}|$. Clearly,

$$\sum_{t \in T} r_t(J) = \binom{m-j}{g-j}. \tag{15}$$

Counting the number of pairs (R, S) with $R \in [X^*]^g$, $J \subset R$, $\Delta(R) = t$ and $S \in [R]^{g-1}$, $J \subseteq S$ in two different ways, (14) implies

$$r_t(J) \cdot (g-j) \le g \cdot \binom{m-j}{g-1-j},$$

hence,

$$r_t(J) \le \frac{g}{g-j} \cdot \binom{m-j}{g-1-j}. \tag{16}$$

For fixed $J \in [X^*]^j$ let $b_j(J)$ denote the number of *bad j-pairs* $\{G, G^*\}$, $G, G^* \in [X^*]^g$, with $J = G \cap G^*$. By (15) and (16) it follows that

$$\begin{aligned} b_j(J) &\le \sum_{t \in T} \binom{r_t(J)}{2} \\ &\le \frac{\binom{m-j}{g-j}}{\frac{g}{g-j} \cdot \binom{m-j}{g-1-j}} \cdot \binom{\frac{g}{g-j} \cdot \binom{m-j}{g-1-j}}{2} \\ &\le \frac{g}{2 \cdot ((g-j)!)^2} \cdot m^{2g-2j-1}. \end{aligned} \tag{17}$$

Now, for $j = 0, 1, \ldots, g-1$ we form $(2g-j)$-uniform hypergraphs $\mathcal{H}_j = (X^*, \mathcal{E}_j)$ with vertex set X^* and edge set $\mathcal{E}_j \subseteq [X^*]^{2g-j}$ as follows: the set $\mathcal{E}_j$ consists of all bad j-pairs, i.e. $E \in \mathcal{E}_j$ if and only if there exist $G, G^* \in [X^*]^g$ with $G \cup G^* = E$ and

$\Delta(G)=\Delta(G^*)$. We will show in the following that there exists a subset Z of X^* of size $c(g)\cdot(m\cdot\log m)^{\frac{1}{2g-1}}$, which is independent for $\mathcal{H}_0,\mathcal{H}_1,\dots,\mathcal{H}_{g-1}$. Then Z is totally multicolored.

By (17) we have

$$(18)\qquad |\mathcal{E}_j|=\sum_{J\in[X^*]^j} b_j(J)\le\binom{m}{j}\cdot\frac{g}{2\cdot((g-j)!)^2}\cdot m^{2g-2j-1}\le c_j^*\cdot m^{2g-j-1}$$

for $j=0,1,\dots,g-1$ and positive constants c_j^*.

Next we will count the number of 2-cycles in $\mathcal{H}_0$. Let $s_{2,i}(\mathcal{H}_0)$ denote the number of $(2,i)$-cycles in $\mathcal{H}_0$, that is the number of unordered pairs $\{E,E^*\}$ with $E,E^*\in\mathcal{E}_0$ and $|E\cap E^*|=i$. Fix an edge $E\in\mathcal{E}_0$ and fix sets $G,G^*\in[X^*]^g$ with $\Delta(G)=\Delta(G^*)$ and $G\cup G^*=E$. Moreover, fix nonnegative integers i_0,i_1. We will count the number $A_{i_0,i_1}(E)$ of unordered pairs $\{H,H^*\}$, where $H,H^*\in[X^*]^g$, with $H\cap H^*=\emptyset$ and $\Delta(H)=\Delta(H^*)$ and $|H\cap G|=i_0$ and $|H^*\cap G^*|=i_1$. There are at most $\binom{g}{i_0}\cdot\binom{m-g}{g-i_0}$ possibilities to choose the set H. Having fixed H with say $\Delta(H)=t$, there are at most $\sum_{I\in[G^*]^{i_1}} r_t(I)$ possibilities to choose H^*.

By (16) we infer for $0\le i_0\le g$ and $0\le i_1\le g-1$ that

$$A_{i_0,i_1}(E)\le\binom{g}{i_0}\cdot\binom{m-g}{g-i_0}\cdot\binom{g}{i_1}\cdot\frac{g}{g-i_1}\cdot\binom{m-i_1}{g-1-i_1}.$$

For $i=2,3,\dots,2g-1$ set

$$A_i(E)=\sum_{i_0+i_1=i} A_{i_0,i_1}(E).$$

Then for $0\le i_0\le g$ and $0\le i_1\le g-1$ we have that

$$A_i(E)\le\sum_{i_0+i_1=i}\binom{g}{i_0}\cdot\binom{m-g}{g-i_0}\cdot\binom{g}{i_1}\cdot\frac{g}{g-i_1}\cdot\binom{m-i_1}{g-1-i_1}\le c_i\cdot m^{2g-1-i},$$

where c_i is a positive constant. With (18) this yields the following upper bound on the number of $(2,i)$-cycles in $\mathcal{H}_0$:

$$(19)\qquad s_{2,i}(\mathcal{H}_0)\le c_i'\cdot m^{2g-1-i}\cdot|\mathcal{E}_0|\le C_i\cdot m^{4g-2-i}$$

for $i=2,3,\dots,2g-1$, where C_i is a positive constant.

Now choose at random vertices from X^* with probability

$$p=m^{-\frac{2g-2}{2g-1}+\epsilon},$$

where ϵ is a positive constant with $\epsilon<\frac{1}{(2g-1)(4g-3)}$, and the vertices are chosen independently of each other.

Let Y be the corresponding random subset of X^*. Then the expected values $E(\cdot)$ satisfy:

$$(20)\qquad E(|Y|)=pm$$

and by (18), for $j=1,2,\dots,g-1$, we obtain that

$$E\left(|\mathcal{E}_j \cap [Y]^{2g-j}|\right) = p^{2g-j}\cdot|\mathcal{E}_j| \le c_j^* pm\cdot\left(p^{2g-j-1}\cdot m^{2g-j-2}\right)$$

$$= c_j^* pm\cdot m^{-\frac{j}{2g-1}+\epsilon(2g-j-1)} = o(pm) \tag{21}$$

for $\epsilon<\frac{1}{(2g-2)(2g-1)}$.

Next, we estimate the expected number $E(s_{2,i})$ of $(2,i)$-cycles in the subhypergraph of $\mathcal{H}_0$ induced on Y. By (19), for $i=2,3,\dots,2g-1$, we have

$$E(s_{2,i}) = p^{4g-i}\cdot s_{2,i}(\mathcal{H}_0) \le C_i pm\cdot\left(p^{4g-i-1}\cdot m^{4g-3-i}\right)$$

$$= C_i pm\cdot\left(m^{-\frac{i-1}{2g-1}+\epsilon(4g-i-1)}\right) = o(pm) \tag{22}$$

provided $\epsilon<\frac{1}{(2g-1)(4g-3)}$.

By (20), (21) and (22), using Chernoff's and Markov's inequality, we infer that there exists a subset $Y\subseteq X^*$ with $|Y|=(1-o(1))\cdot pm$, such that the subhypergraphs of $\mathcal{H}_j$, $j=1,2,\dots,g-1$, induced on Y have $o(pm)$ edges and that $|[Y]^{2g}\cap E_0|\le 2\cdot p^{2g}\cdot|\mathcal{E}_0|$. Moreover, the subhypergraph of $\mathcal{H}_0$ induced on Y has $o(pm)$ $(2,i)$-cycles for $i=2,3,\dots,2g-1$, hence $o(pm)$ 2-cycles. Deleting one vertex from each edge in $\mathcal{E}_j\cap[Y]^{2g-j}$ for $j=1,2,\dots,g-1$, and also from each 2-cycle of $\mathcal{H}_0$ contained in Y, we obtain a subset $Y^*\subseteq Y$ with $|Y^*|=(1-o(1))pm$, which contains no edges from $\mathcal{E}_j$ for $j=1,2,\dots,g-1$, with $|[Y^*]^{2g}\cap\mathcal{E}_0|\le 2\cdot p^{2g}\cdot|\mathcal{E}_0|$ and the subhypergraph $\mathcal{H}^*$ of $\mathcal{H}_0$ induced on Y^* contains no 2-cycles. By deleting all vertices of $\mathcal{H}^*$ of degree bigger than $\frac{8gp^{2g}|\mathcal{E}_0|}{pm}$ we obtain a subset $\overline{Y}\subseteq Y^*$ with $|\overline{Y}|\ge(1-o(1))\frac{pm}{2}$ such that the subhypergraph $\mathcal{H}'$ of $\mathcal{H}^*$ induced on $\overline{Y}$ satiesfies the assumptions of Theorem 4.3 with maximum degree $D_{\mathcal{H}'}\le t^{2g-1}=8c_0^* gp^{2g-1}m^{2g-2}$ by (18).

By (13) we infer that

$$\alpha(\mathcal{H}_0)\ge\alpha(\mathcal{H}')\ge c_{2g}\cdot\frac{pm(1/2-o(1))}{(8c_0^*g)^{\frac{1}{2g-1}}pm^{\frac{2g-2}{2g-1}}}\left(\log\left((8c_0^*g)^{\frac{1}{2g-1}}pm^{\frac{2g-2}{2g-1}}\right)\right)^{\frac{1}{2g-1}}$$

$$\ge c(g)\cdot(m\cdot\log m)^{\frac{1}{2g-1}}.$$

This implies the existence of a totally multicolored subset $Z\subseteq X^*$ with $|Z|\ge c(g)\cdot(m\cdot\log m)^{\frac{1}{2g-1}}$, where $c(g)$ is a positive constant. ∎

We apply Claim 4.2 and Lemma 4.1 to X^* and the coloring Δ^*, where $|I|=g$. Using

$$m^*=\left\lfloor\frac{m-k+1}{k}\right\rfloor$$

$$\ge C_k'\cdot\frac{l^{2k-1}}{\log l},$$

where C'_k is by assumption sufficiently large, say $C'_k \cdot c(g) \geq 1$ and $C'_k \geq 1$ for $g = 1,2,\ldots,k$, and with

$$\left(c(g) \cdot \frac{C'_k \cdot l^{2k-1}}{\log l} \cdot \log \left(\frac{C'_k \cdot l^{2k-1}}{\log l} \right) \right)^{\frac{1}{2g-1}} \geq l$$

for $l \geq l_0(k)$, we obtain an l-element subset $Z \subseteq X^*$, which is totally multicolored with respect to Δ^*. By Claim 4.1 it follows

$$\Delta(S) = \Delta(T) \qquad \text{iff} \qquad S : I = T : I$$

for all $S,T \in [Z]^k$.

Case 2: $C = P$ Let X be monochromatic with respect to the coloring $\hat{\Delta}$. We will show that $|X| \leq k+1$. For contradiction, assume in the following that $|X| \geq k+2$. First we derive some properties of the set P. Recall that $(i,j) \in P$ with $i < j$ implies that for every $(k+1)$-element subset $Z = \{z_1, z_2, \ldots, z_{k+1}\}_< \in [X]^{k+1}$ it is valid that $\Delta(Z \setminus \{z_i\}) = \Delta(Z \setminus \{z_j\})$. As we are dealing with an equivalence relation, we have:

Claim 4.3. *Let h,i,j be positive integers with $1 \leq h < i < j \leq k+1$. Then*

$$\begin{array}{lll} (h,i),(i,j) \in P & \textit{implies} & (h,j) \in P \\ (h,i),(h,j) \in P & \textit{implies} & (i,j) \in P \\ (h,j),(i,j) \in P & \textit{implies} & (h,i) \in P. \end{array}$$

Claim 4.4. *Let i,j be positive integers with $1 \leq i < j \leq k+1$. Then*

$$(i,j) \in P \qquad \textit{implies} \qquad (i,i+1),(j-1,j) \in P.$$

Proof. We show that $(i,i+1) \in P$, the proof for $(j-1,j) \in P$ is similar. Let $Z = \{z_1, z_2, \ldots, z_{k+2}\}_< \in [X]^{k+2}$ be a $(k+2)$-element subset of X. As $(i,j) \in P$, it follows for the sets $Z \setminus \{z_{i+1}\}$ and $Z \setminus \{z_i\}$ that

$$\begin{aligned} \Delta(Z \setminus \{z_i, z_{i+1}\}) &= \Delta(Z \setminus \{z_{i+1}, z_{j+1}\}) \\ \Delta(Z \setminus \{z_i, z_{i+1}\}) &= \Delta(Z \setminus \{z_i, z_{j+1}\}), \end{aligned}$$

hence

$$\Delta(Z \setminus \{z_{i+1}, z_{j+1}\}) = \Delta(Z \setminus \{z_i, z_{j+1}\})$$

implies $(i,i+1) \in P$. ∎

Claim 4.5. *Let i,j be positive integers with $1 \leq i < j \leq k+1$. Then*

$$(i,j) \in P \qquad \textit{implies} \qquad (i^*,j^*) \in P \qquad \textit{for all } i \leq i^* < j^* \leq j.$$

Proof. By Claim 4.4, $(i,j) \in P$ implies $(i,i+1),(j-1,j) \in P$. By Claim 4.3 this gives $(i+1,j),(i,j-1) \in P$. By induction, it follows that $(h,h+1) \in P$ for every positive integer h with $i \leq h < j^*$, and also $(i,i^*),(j^*,j) \in P$, hence again by Claim 4.3 we have $(i^*,j^*) \in P$. ∎

Claim 4.6. *Let g,h,i,j be positive integers with $g<h<i<j$. Then*

$$(g,i),(h,j) \in P \qquad \textit{implies} \qquad (g,j) \in P.$$

Proof. By Claim 4.5, $(g,i) \in P$ implies $(h,i) \in P$. With Claim 4.3 we infer from $(g,i),(h,i),(h,j) \in P$ that $(g,j) \in P$. ∎

Define a partial ordering $\leq_p$ on P as follows: for pairs $(i,j),(i^*,j^*) \in P$ with $i<j$ and $i^*<j^*$ let

$$(i,j) \leq_p (i^*,j^*) \qquad \text{if and only if} \qquad i^* \leq i \text{ and } j^* \geq j.$$

Let P_{max} be the set of maximal elements of P (with respect to $\leq_p$).

If $(g,h),(i,j)$ are two different elements in P_{max}, then by Claim 4.6 either $h < i$ or $j<g$. Let $J=\{i \in \{1,2,\dots,k\} \mid \text{there exists } (j,j^*) \in P_{\max} \text{ with } j \leq i < j^*\}$. Set $I=\{1,2,\dots,k\} \setminus J$.

By Proposition 4.1 and the definition of the coloring $\hat{\Delta}$ it follows that every set X with $|X| \geq k+2$, which is monochromatic with respect to $\hat{\Delta}$ cannot be monochromatic in color P, i.e. it is monochromatic in some color $I \subseteq \{1,2,\dots,k\}$. But then we are again in Case 1. This finishes the proof of Theorem 4.2. ∎

References

[1] M. AJTAI, J. KOMLÓS, J. PINTZ, J. SPENCER, and E. SZEMERÉDI: Extremal uncrowded hypergraphs, *Journal of Combinatorial Theory Ser. A* **32** (1982), 321–335.

[2] N. ALON: On a conjecture of Erdős, T. Sós and Simonovits concerning anti-Ramsey theorems, *Journal of Graph Theory* **7** (1983), 91–94.

[3] N. ALON, H. LEFMANN, and V. RÖDL: On an anti-Ramsey type result, *Colloquia Mathematica Societatis János Bolyai*, 60. Sets, Graphs and Numbers, Budapest, 1991, 9–22.

[4] L. BABAI: An anti-Ramsey theorem, *Graphs and Combinatorics*, **1** (1985), 23–28.

[5] J. BAUMGARTNER: Canonical Partition Relations, *The Journal of Symbolic Logic* **40** (1975), 541–554.

[6] D. DUFFUS, H. LEFMANN, and V. RÖDL: Shift graphs and lower bounds on Ramsey numbers $r_k(l;r)$, to appear.

[7] R. A. DUKE, H. LEFMANN, and V. RÖDL: On uncrowded hypergraphs, 1992, to appear.

[8] P. ERDŐS: Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[9] P. ERDŐS, and A. HAJNAL: Ramsey-type theorems, *Discrete Applied Mathematics* **25** (1989), 37–52.

[10] P. ERDŐS, A. HAJNAL, and R. RADO: Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hung.* **16** (1965), 93–196.

[11] P. ERDŐS, and L. LOVÁSZ: Problems and results on 3-chromatic hypergraphs and some related questions, in: *Infinite and Finite Sets (A. Hajnal, R. Rado and V. T. Sós, eds.)*, North Holland, Amsterdam, 1975, 609–628.

[12] P. ERDŐS, J. NEŠETŘIL, and V. RÖDL: On some problems related to partitions of edges in graphs, in: *Graphs and other combinatorial topics, Proceedings of the third Czechoslovak Symposium on Graph Theory*, ed. M. Fiedler, Teubner Texte in Mathematik vol. 59, Leipzig, 1983, 54–63.

[13] P. ERDŐS, and R. RADO: A combinatorial theorem, *Journal of the London Mathematical Society* **25** (1950), 249–255.

[14] P. ERDŐS, and R. RADO: Combinatorial theorems on classification of subsets of a given set, *Proceedings London Mathematical Society* **2** (1952), 417–439.

[15] P. ERDŐS, V. T. SÓS, and M. SIMONOVITS: Anti-Ramsey Theorems, in: *Infinite and Finite sets, Proceedings Kolloq. Keszthely, Hungary 1973*, eds. A. Hajnal, R. Rado, V. T. Sós, vol. II, Colloq. Math. Soc. János Bolyai 10, Amsterdam, North Holland, 1975, 657–665.

[16] P. ERDŐS, and J. SPENCER: *Probabilistic methods in combinatorics*, Academic Press, New York, 1974.

[17] P. ERDŐS, and E. SZEMERÉDI: On a Ramsey type theorem, *Periodica Mathematica Hungarica* **2** (1972), 1–4.

[18] R. L. GRAHAM, B. L. ROTHSCHILD, and J. H. SPENCER: *Ramsey Theory*, 2nd edition, Wiley-Interscience, New York, 1989.

[19] H. LEFMANN: A note on Ramsey numbers, *Studia Scientiarum Mathematicarum Hungarica* **22** (1987), 445–446.

[20] H. LEFMANN, and V. RÖDL: On canonical Ramsey numbers for coloring three-element sets, in: *Finite and Infinite Combinatorics in Sets and Logic*, (eds.: N. W. Sauer, R. E. Woodrow, B. Sands), Kluwer 1993, 237–247.

[21] H. LEFMANN, and V. RÖDL: On canonical Ramsey numbers for complete graphs versus paths, *Journal of Combinatorial Theory Ser. B* **58** (1993), 1–13.

[22] R. RADO: Anti-Ramsey Theorems, in: *Finite and Infinite Sets*, eds. Hajnal, Rado, Sós, Coll. Math. Soc. János Bolyai, North Holland, 1975, 1159–1168.

[23] R. RADO: Note on canonical partitions, *Bulletin London Mathematical Society* **18** (1986), 123–126.

[24] F. P. RAMSEY: On a problem of formal logic, *Proceedings London Mathematical Society* **30** (1930), 264–286.

[25] M. SIMONOVITS, and V. T. SÓS: On restricted colorings of K_n, *Combinatorica* **4** (1984), 101–110.

[26] J. SPENCER: Asymptotic lower bounds for Ramsey functions, *Discrete Mathematics* **20** (1977), 69–77.

Hanno Lefmann

Lehrstuhl Informatik II, Universität Dortmund, D-44221 Dortmund, Germany

lefmann@ls2.informatik.uni-dortmund.de

Vojtěch Rödl

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322, USA

rodl@mathcs.emory.edu