INTERSECTION THEOREMS
WITH GEOMETRIC CONSEQUENCES

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In this paper we prove that if \( \mathcal{F} \) is a family of \( k \)-subsets of an \( n \)-set, \( \mu_0, \mu_1, \ldots, \mu_s \) are distinct residues mod \( p \) (\( p \) is a prime) such that \( k \equiv \mu_0 \pmod{p} \) and for \( F \neq F' \in \mathcal{F} \) we have \( |F \cap F'| \equiv \mu_i \pmod{p} \) for some \( i, \ 1 \leq i \leq s \), then \( |\mathcal{F}| \equiv \binom{n}{s} \).

As a consequence we show that if \( R^n \) is covered by \( m \) sets with \( m \leq (1+o(1))(1.2)^n \) then there is one set within which all the distances are realised.

It is left open whether the same conclusion holds for composite \( p \).

1. Introduction

Let \( \mathcal{F} \) be a family of \( k \)-element subsets of \( \{1, 2, \ldots, n\} \), and suppose that \( L=\{l_1, l_2, \ldots, l_s\} \) is a subset of \( \{0, 1, \ldots, k-1\} \).

Let us further suppose that for \( F, F' \in \mathcal{F} \) we have

\[
|F \cap F'| \in L.
\]

Ray-Chaudhuri and Wilson [18] proved that (1) implies

\[
|\mathcal{F}| \equiv \binom{n}{s}.
\]

Deza, Erdős and Frankl [2] proved that for \( n \geq n_0(k) \), (2) can be improved to

\[
|\mathcal{F}| \equiv \prod_{i=1}^{s} \frac{n-l_i}{k-l_i}.
\]

In this paper we prove

**Theorem 1.** Suppose \( \mu_0, \mu_1, \ldots, \mu_s \) are distinct residues modulo a prime \( p \), such that

\[
|F| = k \equiv \mu_0 \pmod{p},
\]

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and for any two distinct \( F, F' \in \mathcal{F} \)

\[
|F \cap F'| \equiv \mu_i \pmod{p} \quad \text{for some } i, \quad 1 \leq i \leq s.
\]

Then

\[
|\mathcal{F}| \leq \binom{n}{s}.
\]

Clearly Theorem 1 generalizes (2). It would be interesting to know whether it holds for composite \( p \) as well. In this direction, we prove only

**Theorem 2.** Let \( q \) be a prime power. Suppose that for \( F, F' \in \mathcal{F} \) we have

\[
|F \cap F'| \equiv k \pmod{q}.
\]

Then

\[
|\mathcal{F}| \leq \binom{n}{q-1}.
\]

Let \( \mathbb{R}^n \) denote \( n \)-dimensional Euclidean space. Let us construct a graph on \( \mathbb{R} \) by connecting two points if and only if their distance is 1. Let \( c(\mathbb{R}^n) \) denote the chromatic number of this graph. The exact value of \( c(\mathbb{R}^n) \) seems to be hard to determine. It is known that \( 4 \leq c(\mathbb{R}^2) \leq 7 \). Erdős conjectured that \( c(\mathbb{R}^n) \) is exponential in \( n \). We prove this conjecture in

**Theorem 3.**

\[
c(\mathbb{R}^n) \geq (1 + o(1))(1.2)^n.
\]

Let \( m(n) \) be the minimum integer \( m \) such that \( \mathbb{R}^n \) can be partitioned into \( m \) sets \( X_1, \ldots, X_m \) such that for \( 1 \leq i \leq m \), there is a real number \( r_i \) with the property that \( d(x, y) = r_i \) for all \( x, y \in X_i \) (\( d(x, y) \) denotes the Euclidean distance, i.e., the length of \( x-y \)).

This problem was first considered by Hadwiger [13, 14] in 1944 and 1945. Raiskii [17] proved \( m(n) \geq n + 2 \). This bound was improved by Larman, Rogers [16], then by Larman [15], and again later by Frankl [8]. However none of the lower bounds is exponential. Larman, Rogers [16] proved that

\[
m(n) \geq (3 + o(1))^n,
\]

and they conjectured that \( m(n) \) is exponential in \( n \). Here we prove this conjecture.

**Theorem 4.**

\[
m(n) \geq (1 + o(1))(1.2)^n.
\]

The statement of Theorem 4 will follow from the proof of Theorem 3 using Theorem 2 of Larman, Rogers [16] which states the following:

If \( s \) is a set of \( M \) points in \( \mathbb{R}^n \) with critical distance 1 and critical number \( D \) (i.e., every subset of \( s \) of cardinality exceeding \( D \) contains 2 points at distance 1), then

\[
m(n) \geq M/D.
\]

We prove as well a modification (Conjecture 2 of Larman, Rogers [16]):
Theorem 5. Let $T$ be a set of $m$ vectors in $\mathbb{R}^n$

$$y^{(i)} = (y_1^{(i)}, y_2^{(i)}, \ldots, y_n^{(i)}); \quad i = 1, \ldots, m,$$

with

$$y_j^{(i)} = \pm 1, \quad i = 1, \ldots, m;$$

$y_j^{(i)} = \pm 1$ for $\frac{n}{2}$ values of $1 \leq j \leq n$, such that none of the scalar products $\langle y^{(i)}, y^{(j)} \rangle$ is zero. Then for $n = 4p^x$ ($p$ prime, $x \geq 1$) we have

$$m \equiv 2 \left( \frac{n-1}{n-1} \right) \equiv (1 + o(1))2^n/(1.13)^n. \quad (13)$$

Let $B$ denote the boundary of the unit sphere in $\mathbb{R}^n$ centered at the origin. Let $E$ be a measurable subset of $B$. H. S. Witsenhausen asked for the value of the supremum of the ratio of the measures of $E$ and $B$, assuming that $E$ does not contain two points $A_1, A_2$ which subtend an angle of $90^\circ$ with the center of the sphere. Let $s(n)$ denote this supremum. Choosing $E_0 = \{ y \in B : y_i > 0, \ i = 1, \ldots, n \} \cup \cup \{ y \in B : y_i < 0, \ i = 1, \ldots, n \}$ we see that

$$s(n) \equiv 2^{-n+1}. \quad (14)$$

We prove

Theorem 6.

$$s(n) \leq (1 + o(1))(1.13)^{-n}. \quad (15)$$

For $n > k > l \geq 0$, let $m(n, k, l)$ denote the maximum number of $k$-subsets of an $n$-set such that no two of them intersect in $l$-elements. Erdős [5] conjectured that for $n \geq n_0(k), k \geq 4$, we have

$$m(n, k, l) \leq \max \left\{ \binom{n-l-1}{k-l-1}, \binom{n}{l}/\binom{k}{l} \right\}. \quad (16)$$

Here $\binom{n-l-1}{k-l-1}$ corresponds to all the $k$-subsets containing a fixed $(l+1)$-set while $\binom{n}{l}/\binom{k}{l}$ would correspond to a $(n, k, l)$-Steiner system. In the first case all the intersections have cardinality greater than $l$, in the second smaller than $l$. Frankl [8] proved that for $k \geq 3l+2$

$$m(n, k, l) \leq (1 + o(1))\binom{n-l-1}{k-l-1}. \quad (17)$$

Here we prove

Theorem 7. If $k-l$ is a power of a prime and

(a) $k \geq 2l+1$, then

$$m(n, k, l) = (1 + o(1))\binom{n-l-1}{k-l-1}; \quad (18)$$
(b) $k \leq 2l + 1$, setting $d = 2l - k + 1$ we have

$$m(n, k, l) \geq \binom{n}{d} \binom{n-d}{k-d} = O\left(\binom{n}{l}\right).$$

Let $r(k)$ denote the minimum $n$ such that every graph on $n$ vertices contains either a complete or an empty subgraph on $k$ vertices. Erdős [6] proved

$$r(k) \geq 2^{k/2}.$$  

His proof is probabilistic and in [7] he asked for a constructive bound yielding $r(k) \geq k^t$ for every $t$ for $k \geq k_0(t)$. Such a construction was given in [9].

Here we use Theorem 2 to give a more accurate construction, though still far from the bound (20) (see Theorem 8).

Let $f(n, k, 2)$ denote the maximum cardinality of a collection of $\binom{k}{2}$-subsets of an $\binom{n}{2}$-set such that all the pairwise intersections have for cardinality $\binom{i}{2}$ for $i = 1, 2, \ldots, k-1$.

For $F \subseteq \{1, 2, \ldots, n\}$ set $F(2) = \{\{x, y\}: x \neq y, x, y \in F\}$,

$$\mathcal{G} = \{F(2): F \subseteq \{1, 2, \ldots, n\}, |F| = k\}.$$  

Then $\mathcal{G}$ shows that

$$f(n, k, 2) \geq \binom{n}{k}.$$ 

Frankl [10] conjectured that for $n \geq n_0(k), k \geq 10$ we have equality in (21). Here we prove

**Theorem 9.** If $p$ is an odd prime then we have

$$f(n, p, 2) \geq \binom{n}{2} \cdot \binom{n}{p} \cdot \binom{p-1}{2}.$$  

In [11] it is conjectured that if $\mathcal{F}$ is a collection of 7-element subsets of an $n$-set such that all the pairwise intersections have cardinality 0, 2, 3, 5 or 6 then $|\mathcal{F}| = O(n^2)$. We prove

**Theorem 10.** Let $\mathcal{F}$ be a collection of 7-subsets of an $n$-set, such that for $F, F' \in \mathcal{F}$ we have

$|F \cap F'| \in \{0, 2, 3, 5, 6\}$.

Then

$$|\mathcal{F}| < \binom{n}{2}.$$
In the last paragraph we mention some possible extensions of Theorem 1. In particular we prove:

**Theorem 11.** Suppose \( 0 \leq l_1 < l_2 < \ldots < l_s < n \) are integers and \( \mathcal{F} \) is a collection of subsets of \( \{1, 2, \ldots, n\} \) such that for \( F \neq F' \in \mathcal{F} \) we have

\[
|F \cap F'| \in \{l_1, \ldots, l_s\}.
\]

Then

\[
|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n}{i}.
\]

Note that we do not assume anything about \(|F|\).

2. The proof of Theorem 1

Let \( A_1, A_2, \ldots, A_{\binom{n}{i}} \) be all the \( j \)-subsets and \( B_1, B_2, \ldots, B_{\binom{n}{j}} \) be all the \( i \)-subsets of \( \{1, 2, \ldots, n\} \) with \( j > i \).

Let us define the \( \binom{n}{i} \) by \( \binom{n}{j} \) matrix \( N(i, j) \) in the following way: the 

\((u, v)\)-entry is 1 if \( B_u \subseteq A_v \) and 0 if \( B_u \notin A_v \) for \( 1 \equiv u \equiv \binom{n}{i}, \ 1 \equiv v \equiv \binom{n}{j} \).

For \( i = s, j = k \) let the row-vectors be \( v_1, v_2, \ldots, v_{\binom{n}{s}} \). They are all vectors in \( \mathbb{R}^{\binom{n}{k}} \). Let \( V \) denote the vector space generated by the \( v_i \)'s, \( 1 \equiv i \equiv \binom{n}{s} \). Obviously we have

\[
\text{dim } V \equiv \binom{n}{s}.
\]

The following identity can be checked easily \( (0 \equiv i < s) \)

\[
N(i, s)N(s, k) = \binom{k-i}{s-i} N(i, k).
\]

Consequently, for \( 0 \equiv i < s \), the row vectors of \( N(i, k) \) are contained in \( V \). Let us count the product \( N(i, k)^TN(i, k) = M(i, k) \), where \( N^T \) denotes the transpose of \( N \). Of course \( M(i, k) \) is an \( \binom{n}{k} \) by \( \binom{n}{k} \) matrix in which the \((u, v)\),

\[
\text{entry is } \binom{|A_u \cap A_v|}{i} \text{ for } 1 \equiv u, v \equiv \binom{n}{k}. \text{ Moreover the row-vectors of } M(i, k) \text{ are linear combinations of the rows of } N(i, k), \text{ and consequently they are contained in } V.
\]

Let us choose \( 0 \equiv a_i \leq p \) for \( 0 \equiv i \leq s_0 \) in such a way that for every integer \( x \) we have

\[
\prod_{i=1}^{s_0} (x - \mu_i) \equiv \sum_{i=1}^{s_0} a_i \binom{x}{i} \pmod{p}.
\]
Let us set \( M = \sum_{i=1}^{s} a_i M(i, k) \), where the addition is to be done componentwise, i.e., in position \((u, v)\) of \( M \) we have

\[
M(u, v) = \sum_{i=1}^{s} a_i \left( |A_u \cap A_v| \right).
\]

By the definition of \( M \) the row-vectors of \( M \) are in \( V \), and consequently (23) gives:

\[
\text{rank } M \equiv \text{dim } V \equiv \binom{n}{s}.
\]

Now let \( M(\mathcal{F}) \) be the minor spanned by the elements \( m(u, v) \) for which \( A_u, A_v \in \mathcal{F} \).

The assumptions of the theorem and (25) and (26) yield that for \( A_u, A_v \in \mathcal{F}, u \neq v \), we have

\[
m(u, v) \equiv 0 \pmod{p}
\]

and

\[
m(u, u) \not\equiv 0 \pmod{p}.
\]

Consequently the determinant of \( M(\mathcal{F}) \) is not congruent to 0 modulo \( p \), whence \( \det M(\mathcal{F}) \neq 0 \). Thus using (27) we infer

\[
|\mathcal{F}| = \text{rank } M(\mathcal{F}) \equiv \text{rank } M \equiv \binom{n}{s}.
\]

Now we prove Theorem 2. We need an easy lemma.

**Lemma.** Let \( q = p^x \), \( p \) is a prime, \( x \geq 1 \). Then for \( a \equiv p \left( \frac{a}{q-1} \right) \) if and only if \( a \equiv -1 \pmod{q} \).

The proof of the lemma is elementary and we leave it to the reader.

Let us choose real numbers \( a_i, 0 \leq i < q \), such that

\[
\sum_{i=0}^{q-1} a_i \binom{x}{i} = \binom{x-k-1}{q-1}.
\]

Then by the lemma all the off-diagonal entries are zero mod \( p \) in the minor corresponding to \( \mathcal{F} \) of the matrix \( M = \sum_{i=0}^{q-1} a_i M(i, k) \), but the diagonal entries are non-zero mod \( p \) consequently the minor is again of full rank, yielding

\[
|\mathcal{F}| \equiv \text{rank } M \equiv \binom{n}{q-1}.
\]

**Remark.** The critical distance \( \alpha \) of the method yields is expected to be

\[
\left( \begin{array}{c}
\text{any} \\
\text{most}
\end{array} \right)
\]

(a) Since which

(b) For a

Hence

\[
(28)
\]
3. The proof of Theorems 3 and 4

Let us consider the set $S$ of vectors $x=(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ for which $x_i=0$ \((n-2q+1)\)-times and $x_i=1/\sqrt{2q}$ the remaining \((2q-1)\) times. Then

$$|S| = \binom{n}{2q-1}.$$

Let us associate with $v \in S$ the \((2q-1)\)-set $F(v)=\{i : x_i \neq 0\}$. Then obviously $d(x, y)=1$ is equivalent to $|F(x) \cap F(y)|=q-1$. Thus by Theorem 2 among any $\binom{n}{q-1}+1$ vectors in $S$ there are two at distance 1, i.e., every color contains at most $\binom{n}{q-1}$ of them, yielding

$$c(\mathbb{R}^n) = \max_{q \text{ is a prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}}.$$

Now choosing $q$ to be $(1+o(1))\frac{2-\sqrt{2}}{2}n$ we obtain

$$c(\mathbb{R}^n) \approx (1+o(1))(1.2)^n.$$

Remark. Since for $q=2^{2l+1}$ the expression $1/\sqrt{2q}=2^{-l-1}$ is rational, the same method yields that the chromatic number of the set of vectors with rational coordinates is exponential as well.

The statement of Theorem 4 follows now from the fact that the set $S$ has critical distance 1 and critical number $\binom{n}{q-1}$ (cf. the introduction).

4. The proof of Theorem 7

(a) Since $k \equiv 2l+1$ then $k-l \geq l$. Thus $l$ is the only integer between 0 and $k-1$ which is congruent to $k \pmod{q} = k \pmod{(k-l)}$. We can apply Theorem 2, and obtain

$$m(n, k, l) \equiv \binom{n}{k-l-1} = (1+o(1))\binom{n-l-1}{k-l-1},$$

proving (18).

(b) For a $d$-subset $D$ of \(\{1, 2, \ldots, n\}\) let $\mathcal{G}(D)$ be the collection of those members of the family which contain $D$. Of course

$$\sum_D |\mathcal{G}(D)| = m\left(\begin{array}{c} k \\ d \end{array}\right).$$

Hence we can choose $D_0$ such that

$$|\mathcal{G}(D_0)| \geq m\left(\begin{array}{c} k \\ d \end{array}\right)/\left(\begin{array}{c} n \\ d \end{array}\right).$$
Set $\mathcal{F} = \{G - D_0 : G \in \mathcal{G}(D_0)\}$. Then $\mathcal{F}$ is a family of $(k - d)$-subsets of the $(n-d)$-set $\{1, 2, \ldots, n\} - D$, no two of which intersect in $l - d$ elements. Since $k - l > l - d$ we can apply Theorem 2, which gives

$$|\mathcal{F}| \equiv \binom{n-d}{k-l-1} = \binom{n-d}{l-d}.$$  

From (28) and (29) we obtain

$$m(n, k, l) \equiv \binom{n}{d} / \binom{k}{d} \binom{n-d}{l-d} = O\left(\binom{n}{l}\right).$$

5. The proof of Theorem 5 and Theorem 6

Let us define $F_i = \{j : y_j^{(0)} = +1\}$. Then $|F_i| = 2p^x$, and the condition implies $|F_i \cap F_{i'}| \neq p^x$.

Now apply Theorem 7 with $k = 2p^x$, $l = p^x$, $d = 1$, and deduce

$$m \equiv 2\left(\frac{4p^x-1}{p^x-1}\right) \equiv (1 + o(1))2^x/(1.13)^n.$$  

To prove Theorem 6 we choose $q$ to be the smallest prime power which is at least $n/4$. Let $\alpha$, $\beta$ be two real numbers and let $S(\alpha, \beta)$ be the set of vectors $y = (y_1, y_2, \ldots, y_n)$ for which

$$y_i = \alpha \quad (2q - 1) \text{ times, and } y_i = \beta \quad (n - 2q + 1) \text{ times.}$$

For $y \in S(\alpha, \beta)$ set $F(y) = \{i : y_i = \alpha\}$. Now the length of $y$ is $\sqrt{(2q-1)\alpha^2 + (n-2q+1)\beta^2}$, i.e., $y$ is on $B$ iff

$$\text{(30)} \quad (2q-1)\alpha^2 + (n-2q+1)\beta^2 = 1.$$

If $|F(y) \cap F(y')| = q-1$ then

$$\langle y, y' \rangle = (q-1)\alpha^2 + (n-3q+1)\beta^2 + 2q\alpha\beta.$$

To make this scalar product vanish we need

$$\text{(31)} \quad (q-1)\alpha^2 + (n-3q+1)\beta^2 + 2q\alpha\beta = 0.$$

Since $q \equiv \frac{n}{4}$ the system (30), (31) is solvable in real $\alpha$, $\beta$. Let $S$ be the image of $S(\alpha, \beta)$ under any orthogonal transformation of $B$. Then $|S| = |S(\alpha, \beta)| = \binom{n}{2q-1}$, and applying Theorem 2 with $k = 2q - 1$, the special choice above of $\alpha, \beta$ gives:

$$\text{(32)} \quad \frac{|E \cap S|}{|B \cap S|} = \frac{|E \cap S|}{|S|} \equiv \frac{n}{q-1} \equiv (1 + o(1))(1.13)^{-n}.$$
Now averaging over the orthogonal group yields
\[
\frac{\mu(E)}{\mu(B)} \leq \max_S \frac{|E \cap S|}{|S|} \leq (1 + o(1))(1.13)^{-n},
\]
yielding (15).

6. Constructive Ramsey-bound

**Theorem 8.** Let us set \( V(\mathcal{G}) = \{F \subseteq \{1, 2, \ldots, n\}: |F| = q^2 - 1\} \), \( q \) is a prime power, and \( E(\mathcal{G}) = \{\langle F, F' \rangle: |F \cap F'| \not\equiv -1 (\text{mod } q)\} \).

Then \( \mathcal{G} \) contains no complete or empty subgraph on more than \( \binom{n}{q-1} \) vertices.

**Proof.** If \( F_1, \ldots, F_m \) is a complete subgraph then \( |F_i \cap F_j| \equiv -1 (\text{mod } q) \) for every \( 1 \leq i < j \leq m \). Thus Theorem 2 gives the assertion.

If \( F_1, \ldots, F_m \) is an empty subgraph then \( |F_i \cap F_j| \in \{q - 1, 2q - 1, \ldots, q^2 - q - 1\} \) for \( 1 \leq i < j \leq m \), thus (2) gives the statement.

Setting \( n = p^2 \), \( q = p \), we obtain
\[
r(k) \geq \exp\left((1 + o(1)) \log^2 k / 4 \log \log k\right).
\]

7. The proof of Theorems 9 and 10

Let \( x \) be a point of maximal degree and set
\[
\mathcal{F}_0 = \{F \in \mathcal{F}: x \in F\}.
\]

Then
\[
|\mathcal{F}_0| \geq |\mathcal{F}| \left(\begin{array}{c} p \\ 2 \end{array}\right) / \left(\begin{array}{c} n \\ 2 \end{array}\right),
\]
and for \( F, F' \in \mathcal{F}_0 \) we have
\[
|F \cap F'| \in \left\{\left(\begin{array}{c} 2 \\ 2 \end{array}\right), \left(\begin{array}{c} 3 \\ 2 \end{array}\right), \ldots, \left(\begin{array}{c} p - 1 \\ 2 \end{array}\right)\right\}.
\]

Since \( \left(\begin{array}{c} i \\ 2 \end{array}\right) - \left(\begin{array}{c} p - i + 1 \\ 2 \end{array}\right) = \frac{(2i - 1)p - p^2}{2} \equiv 0 (\text{mod } p) \), and \( p \left(\begin{array}{c} i \\ 2 \end{array}\right) \) for \( i = 2, \ldots, p - 1 \), the intersections lie in \( \frac{p - 1}{2} \) different non-zero congruence classes modulo \( p \). On the other hand \( p \left(\begin{array}{c} p \\ 2 \end{array}\right) = |F| \), and therefore Theorem 1 yields
\[
|\mathcal{F}_0| \leq \left\{\left(\begin{array}{c} n \\ 2 \end{array}\right), \left(\begin{array}{c} p - 1 \\ 2 \end{array}\right)\right\}.
\]

Now (33) and (34) imply (22).

Theorem 10 is an immediate consequence of Theorem 1: Simply set \( k = 7, \mu_0 = 1, \mu_1 = 0, \mu_2 = 2, p = 3 \).
8. On possible extensions

First we prove Theorem 11.

Let $F_1, F_2, \ldots, F_m$ be the sets in our family arranged so that $|F_1| \equiv |F_2| \equiv \ldots \equiv |F_m|$. For $0 \leq i \leq s$, let $A_1, \ldots, A^{(n)}$ be the different $i$-subsets of $\{1, 2, \ldots, n\}$.

Let $N(i)$ be the $m$ by $\binom{n}{i}$ matrix which has 1 or 0 in the position $(u, v)$ according to whether $A_u \subseteq F_v$ or not, $1 \leq u \leq m$, $1 \leq v \leq \binom{n}{i}$. Of course $r(N(i)) \leq \binom{n}{i}$.

Let us set $M(i) = N(i)N(i)^T$. Then $M(i)$ is $m$ by $m$ with $|F_u \cap F_v|$ in position $(u, v)$, and we still have

$$r(M(i)) \leq \binom{n}{i}.$$ 

Let $u_1^{(i)}, \ldots, u_m^{(i)}$ be the row-vectors of $M(i)$, and let $V$ be the vector space spanned by the $u_j^{(i)}$ for $1 \leq i \leq s$, $1 \leq j \leq m$. Then we have

$$\dim V = \sum_{i=0}^{s} r(M(i)) \leq \sum_{i=0}^{s} \binom{n}{i}. \tag{35}$$

Let us choose $a_v^{(i)}$ for fixed $i$, $1 \leq i \leq s$, and $v = 0, 1, \ldots, i$ that

$$\sum_{v=0}^{i} a_v^{(i)} \binom{x}{v} = \prod_{i=1}^{s} (x - l_i). \tag{36}$$

Now we define an $m$ by $m$ matrix $M$. If $1 \leq u \leq m$ and $i$ is the greatest integer for which $|F_u| > l_i$, then let the $u$th row of $M$ be

$$\sum_{v=0}^{i} a_v^{(i)} u_v^{(v)}. \tag{37}$$

If $u = m$, and $|F_u| = l_s$, then the last row of $M$ is $v_m^{(i)}$. Since all the row-vectors are in $V$ we have by (35)

$$r(M) \leq \sum_{i=0}^{s} \binom{n}{i}. \tag{38}$$

By (36) and (37) the $u$'th diagonal entry of $M$ is

$$\prod_{i=1}^{s} (|F_u| - l_i) \neq 0, \text{ since } |F_u| > l_i.$$ 

Since $|F_u| \equiv |F_v|$ for $u \neq v$, in this case $|F_u \cap F_v| \in \{l_1, l_2, \ldots, l_s\}$, and consequently by (26) and (37) the $(u, v)$-entry of $M$ is 0. This means that $M$ is lower-triangular with non-zero diagonal consequently of full rank; thus (38) yields

$$|\mathcal{F}| = m = \text{rank } M \leq \prod_{i=0}^{s} \binom{n}{i}. \tag{39}$$

Proof. Choo

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The most important extension is to decide whether Theorem 1 or at least Theorem 2 holds for congruences modulo arbitrary positive integers.

Frankl, Rosenberg [12] proved that for \( s = 1 \) Theorem 1 extends to arbitrary integer moduli (which generalizes results by Ryser [19], Deza, Erdős, Singhi [3], Babai, Frankl [1], and Deza, Rosenberg [4]).

The first open case modulo a prime power is for 8: \( \mu_0 = 0, \mu_1 = 1, \mu_2 = 2, \mu_3 = 4 \) and \( \mu_4 = 6 \).

By the proof of Theorem 1 we can prove

**Theorem 12.** Suppose \( q \) is a power of the prime \( p \). Let \( \mu_0, \mu_1, \ldots, \mu_s \) be distinct residues modulo \( q \). Let \( \mathcal{F} \) be a collection of \( k \)-subsets of \( \{1, 2, \ldots, n\} \), such that for \( F \neq F' \in \mathcal{F} \) we have

\[
|F| \equiv \mu_i \pmod{q},
\]

\[
|F \cap F'| \equiv \mu_i \pmod{q} \quad \text{for some} \quad 1 \leq i \leq s.
\]

If there exists a rational polynomial \( g(x) \) of degree \( d \) such that \( p \mid g(k) \) (\( g(k) \) is an integer) but \( p \nmid g(x) \) for \( x \equiv \mu_i \pmod{q} \), \( i = 1, \ldots, s \), then

\[
|\mathcal{F}| \leq \binom{n}{d}.
\]

**Proof.** Choose the rational numbers \( a_0, a_1, \ldots, a_d \) in such a way that

\[
\sum_{y=0}^{d} a_y \binom{x}{y} = p(x).
\]

Then the matrix \( M = \sum_{y=0}^{d} a_y M(v, k) \) contains a full-rank minor corresponding to the members of \( \mathcal{F} \), yielding

\[
|\mathcal{F}| \leq \text{rank } M \leq \text{rank } M(d, k) \leq \binom{n}{d}.
\]

**References**