Europ. J. Combinatorics (1997) **18,** 1–9

A New Game Chromatic Number

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Consider the following two-person game on a graph G. Players I and II move alternatively to color a yet uncolored vertex of G properly using a pre-specified set of colors. Furthermore, Player II can only use the colors that have been used, unless he is forced to use a new color to guarantee that the graph is colored properly. The game ends when some player can no longer move. Player I wins if all vertices of G are colored. Otherwise Player II wins. What is the minimal number $\chi_g^*(G)$ of colors such that Player I has a winning strategy? This problem is motivated by the game chromatic number $\chi_g(G)$ introduced by Bodlaender and by the continued work of Faigle, Kern, Kierstead and Trotter. In this paper, we show that $\chi_g^*(T) \leqslant 3$ for each tree T. We are also interested in determining the graphs G for which $\chi(G) = \chi_g^*(G)$, as well as $\chi_g^*(G)$ for the k-inductive graphs where k is a fixed positive integer.

1. Introduction

Let $G = (V, E)$ be a graph and let X be a finite set. A vertex coloring of G is a mapping from V to X. A coloring of G is proper if u and v are assigned different colors whenever uv is an edge in G. As usual, the chromatic number is defined to be the smallest $|X|$ such that the graph G can be colored properly with the colors of X.

The game chromatic number was introduced by Bodlaender [1], and developed by Faigle, Kern, Kierstead and Trotter [5]. For the completeness of this paper, we state the definition of game chromatic number given in [5].

The game chromatic number is based on the following modified coloring problem as a two-person game in which the first player tries to color a graph and the second tries to prevent this from happening. Let $G = (V, E)$ be a graph, let t be a positive integer, and let X be a set of colors with $|X| = t$. The two persons compete in a two-person game lasting at most $n = |V|$ moves. They alternate turns, with a move consisting of selection of a previously uncolored vertex x and assigning it a color from X distinct from the colors assigned previously (by either player) to neighbors of X. If, after n, moves, the graph is colored, the first player is the winner. The second player wins if an impasse is reached before all vertices in the graph are colored; that is, if for every uncolored vertex x and every color α from X, x is adjacent to a vertex having color α. The *game chromatic number* of a graph G, denoted by $\chi_g(G)$, is the last t for which the first player has a winning strategy.

The *game chromatic number* of a family $\mathscr{F}$ of graphs, denoted $\chi(\mathscr{F})$, is then defined to be $\max\{\chi_g(G) : G \in \mathscr{F}\}$, provided that the value is finite; otherwise, $\chi_g(\mathscr{F})$ is infinite.

It was shown that the game chromatic number of the family of trees is at least 4 and at most 5 by Bodlaender [1]. Later, Faigle, Fern, Kierstead and Trotter [5] showed that the game chromatic number of the family of trees is 4. Recently, Kierstead and Trotter [6] proved that the game chromatic number of the family of planar graphs is at most 33. Inspired by their work, we present in a new coloring game with one more condition for the second player. The additional condition is that the second player may only use one of the colors introduced earlier by the the first player unless he is forced to use a new color to guarantee that the graph is colored properly. Such a game is called the

0195-6698/97/010001 + 09 $25.00/0 ej940071

chromatic game II. Similarly, the game chromatic number II of a graph G, denoted by $\chi_g^*(G)$, is the least t for which the first player has a winning strategy in the chromatic game number II. The game chromatic number II of a family $\mathscr{F}$, is defined to be $\max\{\chi_g^*(G): G \in \mathscr{F}\}$, provided that the value is finite. Otherwise, $\chi_g^*(\mathscr{F})$ is infinite. Clearly,

$$\chi(G) \leqslant \chi_g^*(G) \leqslant \chi_g(G) \leqslant \Delta(G) + 1,$$

where $\Delta(G)$ is the maximum degree of G.

Let G be a graph obtained from a complete bipartite graph $K_{n,n}$ by deleting a perfect matching. Then, $\chi(G) = 2$ and $\chi_g^*(G) = \chi_g(G) = n$. Thus, there are infinitely many graphs such that $\chi_g^*(G) - \chi(G)$ is unbounded. In the next section, we will investigate the graphs for which $\chi(G) = \chi_g^*(G)$. In Section 3, we will determine $\chi_g^*(T)$ for each tree T. In Section 4, the game chromatic number II is investigated for k-inductive graphs with a fixed constant k.

Let $G = (V, E)$ be a graph and let v be a vertex of G. We will use $N(v)$ to denote the open neighborhood of v and $N[v]$ to denote the closed neighborhood of v; that is, $N[v] = N(v) \cup \{v\}$. In general, for a positive integer m, we define

$$N_m[v] = \{w \in V : \text{the distance between } v \text{ and } w \text{ is at most } m\}.$$

Clearly, $N_1[v] = N[v]$.

2. Graphs for Which $\chi(G) = \chi_g^*(G)$

In this section, we investigate graphs for which the chromatic number and the game chromatic number II are same. First, we characterize all bipartite graphs satisfying the above condition.

Theorem 1. *Let $G = (V, E)$ be a connected bipartite graph with parts V_1 and V_2. Then, $\chi_g^*(G) = 2$ iff there is a vertex $v \in V$ such that $N_2[v] = V$; that is, $N(v) = V_j$ for some $v \in V_i$, where $\{i, j\} = \{1, 2\}$.*

Proof. To prove the sufficiency, without loss of generality, assume that there is a vertex $v_1 \in V_1$ such that $N(v_1) = V_2$. Clearly, it is sufficient to show that $\chi_g^*(G) \leqslant 2$. Let $X = \{1, 2\}$ be a color set. Initially, the first player colors the vertex v_1 with the color 1. Since $N(v_1) = V_2$, by using the color 1, the two players can only color the vertices in V_1. Thus, the first player has a winning strategy if he does not introduce color 2 until all vertices in V_1 are colored.

To prove the necessity, suppose that $N(v_i) \neq V_{i+1}$ if for every $v_i \in V_i$ and $i = 1, 2$, where the index is taken modulo 2. Let $X = \{1, 2\}$ be a color set. Without loss of generality, assume that the first player first colors color a vertex $v_1 \in V_1$. Then, the second player has a winning strategy if he colors a vertex $v_2 \in V_2 - N(v_1)$ with color 1 for his first step. Therefore, $\chi_g^*(G) \geqslant 3$.

In Bodlaender's original game, the coloring number of a graph may depend on whether the cooperative or the obstructive player makes the first move. In the present model, the coloring number of a graph may also depend on whether the cooperative or the obstructive player makes the first move. For instance, let G be a graph obtained from $P_5 = v_1v_2v_3v_4v_5$ by adding a vertex v_0 and an edge v_0v_3. Clearly, $N_2[v_3] = V(G)$ holds. By Theorem 1, $\chi_g^*(G) = 2$ if the cooperative player makes the first move. On the other hand, it is not difficult to see that the coloring number is 3 if the obstructive player makes the first move. In fact, the strategy for the obstructive player is that of coloring the vertex v_0 at the first step and then coloring either v_1 or v_5 with the same color as used for v_0. (This move depends on which vertex of v_1 or v_5 is available after the cooperative player has made his/her first move.)

It is not difficult to see that the 'if' part of the above result can be generalized as follows.

THEOREM 2. *Let G be a m-multipartite graph with parts $V_1, V_2, \ldots, V_m$. If there is a vertex $v_i \in V_i$ such that $N(v_i) = V(G) - V_i$ for each $i = 1, 2, \ldots, m$, then $\chi_g^*(G) = \chi_g(G) = m$.*

Let $K_{l_1,l_2,\ldots,l_m}$ be a complete m-multipartite graph with the vertex set partition $V_1 \cup V_2 \cup \cdots \cup V_m$ such that $|V_i| = l_i$. Then, $\chi_g^*(K_{l_1,l_2,\ldots,l_m}) = m$. If $l_i \geqslant 2$ for every $i = 1, 2, \ldots, m$, $\chi_g(K_{l_1,l_2,\ldots,l_m}) = 2m - 1$. Also, the following result is of interest.

THEOREM 3. *Let P denote the Petersen graph. Then, $\chi(P) = \chi_g^*(P) = 3$ and $\chi_g(P) = 4$.*

PROOF. To prove the theorem, we refer to the outer cycle of the Petersen graph by $v_1v_2v_3v_4v_5v_1$, to the inner cycle by $v_6v_7v_8v_9v_{10}v_6$, and to the matching between two cycles as v_1v_6, v_2v_9, v_3v_7, v_4v_{10} and v_5v_8. The first player initially colors v_1 and only uses color 1 until a maximum independent set has been assigned color 1. Without loss of generality, we assume that the second player will color one of v_3, v_7 and v_9:

(i) If the second player colors v_3, the first player colors v_{10}. Then, the second player has to color v_8 with color 1.

(ii) If the second player colors v_9, the first player colors v_7. Then, the second player has to color v_4 with color 1.

(iii) If the second player colors v_7, the first player colors v_9. Then, the second player has to color v_4 with color 1.

Notice that the graph formed from P by deleting the colored vertices in each case described above is the union of the three independent edges. Therefore, $\chi_g^*(P) = 3$. □

One of the main reasons for introducing $\chi_g^*(G)$ is that it is likely that $\chi_g^*(G) = \chi(G)$ for many graphs G. This likelihood is intriguing in that such graphs G can be properly colored with χ colors by two persons (each coloring essentially one-half of the vertices) with only one of these persons coloring in an intelligent way. We shall determine a class of graphs G where $\chi_g^*(G) = \chi(G)$. It is likely that when G is such that $\chi(G) < \delta(G)$, then $\chi(G) < \chi_g(G)$. Thus one would not expect to find many graphs G for which $\chi_g(G) = \chi(G)$. It is of interest to determine graphs for which the chromatic number and the game chromatic number II are the same. We think that the graphs with bounded degrees and a very large girth may have this property. It also is of interest to determine the graphs for which the two different chromatic numbers are same.

The clique number $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. In general, $\chi(G) \geqslant \omega(G)$ for every graph G. Seinsche [7] proved that the above equality holds if the graph G is P_4 free, where P_4 denotes the path with four vertices. We have obtained the following result.

THEOREM 4. *If a graph G is P_4 free, then $\omega(G) = \chi(G) = \chi_g^*(G)$.*

The proof depends on the following known lemma.

LEMMA 1. *If a graph G is P_4 free, then each maximal independent set in G contains a vertex of each maximal clique.*

PROOF. Suppose that the lemma is false: let H be a maximal independent set and let L be a maximal clique in G such that $H \cap L = \phi$. For each $v \in L$, let $H_v = N_{\bar{G}} \cap H$, i.e. H_v is the non-neighbors of v in H. We show that there are $v_1, v_2 \in V(G)$ such that $H_{v_1} \subsetneq H_{v_2}$ and $H_{v_2} \subsetneq H_{v_1}$. Since L is a maximal clique, there is a vertex $v \in$ L such that $H_v \neq \phi$. Also, since H is a maximal independent set, H_v is a proper subset of H for any $v \in L$. Finally, each $w \in H$ is contained in H_v for some $v \in L$, again since L is a maximal clique. Thus, there are at least two vertices $v_1 \neq v_2$ such that $H_{v_1} \subsetneq H_{v_2}$ and $H_{v_2} \subsetneq H_{v_1}$. Thus pick $w_1 \in H_{v_1} - H_{v_2}$, $w_2 \in H_{v_2} - H_{v_1}$, giving $w_1 v_2 v_1 w_2$ as an induced P_4 in G, a contradiction. □

PROOF OF THEOREM 4. Let the first player and the second player alternately color a maximal independent set I_1 with the color 1. Thus, when color 2 is introduced (by either of the two players) each uncolored vertex is adjacent to some vertex of color 1. Continue to color with color 2, forming a maximal independent set I_2 in $V(G) - I_1$. Thus each vertex of $G - (I_1 \cup I_2)$ is adjacent to a vertex of I_1 and a vertex of I_2. Continue this process, coloring a maximal independent set I_3 in $G - (I_1 \cup I_2)$ with color 3. We eventually color all vertices with, say, t colors. The claim is proved by showing that G contains a clique of order t. It will be done by induction on t, being clear for $t \leqslant 2$. Consider the induced subgraph $G - I_1$ which contains a maximal clique M on $t-1$ vertices (one vertex from each of $I_2, I_3, \ldots, I_t$). By the lemma, M is not a maximal clique in G. Thus, it can be enlarged to a maximal clique in G and, as such, contains a vertex of the maximal independent set I_1 in G. Hence G contains a clique of order t. □

3. TREES

The following theorem completely determines $\chi_g^*(T)$ for all trees.

THEOREM 5. *Let T be a tree with order $|V(T)| \geqslant 2$. Then,*

$$\chi_g^*(T) = \begin{cases} 2 & \text{if } N_2[v] = V(T) \text{ for some vertex } v \in V(T), \\ 3 & \text{otherwise.} \end{cases}$$

PROOF. Theorem 1 shows that

$$\chi_g^*(T) \begin{cases} = 2 & \text{if } N_2[v] = V(T) \text{ for some vertex } v \in V(T), \\ \geqslant 3 & \text{otherwise.} \end{cases}$$

The remainder of the proof will show that $\chi_g^*(T) \leqslant 3$. To do so, let $X = \{1, 2, 3\}$ be a color set. First, we root the tree T and denote the root by r. We will describe an algorithmic way of first coloring all but an independent set of vertices of T with the colors 1 and 2. To begin, the first player colors the root r with color 1.

Since T is a tree, there is an unique path, denoted by $P(u, v)$, from u to v for any pair vertices u and v in T. If $u = r$, for simplicity let $P(u, v) = P(v)$. Sometimes, we slightly vary the notation by using $P(u, v)$ for $V(P(u, v))$.

At any intermediate point in the game, we let C denote the set of colored vertices, U be the set of uncolored vertices, and $U^* \subseteq U$ be the set of vertices every one of which can be assigned to color 1 or 2 such that the resulting coloring is proper.

Suppose that the second player has just colored a vertex v. Let $w \in P(v)$ such that there is a vertex $x \in C$ with $v \notin P(x)$ and $P(x) \cap P(v) = P(w)$ and $d(v, w)$ is a minimum. Clearly, w is well defined, since $r \in C \cap P(v)$. Then, the first player will choose a vertex $u \in U^*$ and color it with a feasible color of either 1 or 2 as follows:

(i) Set $u = w$ if $w \in U^*$. Otherwise, go to step (ii).

(ii) Set u to be the predecessor of w in $P(v)$ if $w \notin U^*$ and the predecessor is in U^*. Otherwise, go to step (iii).

(iii) Set u to be a neighbor of w such that $w \in P(u)$ and $u \in U^*$. In this step, if the successor of w in $P(x)$ is in U^*, then set u to be the successor; if the successor of w in $P(x)$ is not in U^* and the successor of w in $P(x')$ (for some x' such that $P(x') \cap P(v) = P(w)$) is in U^*, set u to be the successor. Otherwise, go to step (iv).

(iv) Let $u \in U^*$ such that $P(u) \cap U^* = \{u\}$.

In the following, we will prove that if $U^* = \phi$ at some step during either player's turn, then U is an independent vertex set.

Suppose, to the contrary, that there are a pair vertices x and y in U and $xy \in E(G)$. Without loss of generality, assume that $y \in P(x)$. For convenience, in the following, let C_1 be the set of vertices which are colored with color 1, and let C_2 be the set of vertices which are colored with color 2. Since $U^* = \varnothing$, $X_1 = N(x) \cap C_1 \neq \varnothing$ and $X_2 = N(x) \cap C_2 \neq \varnothing$. For every $s \in X_1 \cup X_2$, let s^* be the vertex which was colored first in $\{t \in C : s \in P(t)\}$. Let $x_i \in X_i$ be such that x_i^* is the vertex which was first colored among all $\{s^* : s \in X_i\}$ for $i = 1, 2$. Also, note that $Y_1 = N(y) \cap C_1 \neq \varnothing$ and $Y_2 = N(y) \cap C_2 \neq \varnothing$. Define y_1, y_2, y_1^*, and y_2^* in a manner similar to that done above.

Note that x_1^* and x_2^* have been colored by the second player, since we assume that the first player uses the selection rule described above. Without loss of generality, assume that x_1^* was colored before x_2^*. From the first player's selection rule described above, x_2^* was colored before any vertex s^* for any $s \in X_1 - \{x_1\}$. Also, $x_2 = x_2^*$; otherwise x would have been colored by the first player, a contradiction. Since y is not yet colored and y is the predecessor of x, y_1^* and y_2^* must have been colored before x_2^* was colored.

If both y_1^* and y_2^* were colored before x_1^* was colored, then the predecessor of y was either colored or it was impossible for it to be colored properly by using color 1 or color 2. Then, x would have been colored by the first player after the second player colored the vertex x_1^*, which leads to a contradiction.

If one of $\{y_1^*, y_2^*\}$ was colored before x_1^* was colored and the other one is colored after x_1^* is colored, then y would have been colored by the first player after the second player had colored x_1^*, a contradiction. Thus, both y_1^* and y_2^* were colored between x_1^* and x_2^*. Without loss of generality, assume that y_1^* was colored before y_2^*.

If $y \in P(y_1^*)$, then y_1^* was colored by the second player and y would have been colored by the first player, again a contradiction. Thus, $y_1 = y_1^*$ is the predecessor of y in $P(y)$ and $y \in P(y_2)$.

Now, we consider which vertex the first player would color after the second player colored y_2. Notice that y_1 is the predecessor of y. Since it is impossible to color y with either color 1 or color 2 and $P(y_2, y) = y_2 y$, by the selection rule, the first player would pick a vertex $u \in N(y) \cap U^*$ such that $y \in P(u)$. Clearly, x is a qualified candidate for the first player to choose at this step. We will show that x is the only candidate for the first player at this step. To the contrary, suppose that there is another candidate $s \neq x$. Let s^* be a vertex which was first to be colored in the set $\{t : s \in P(t)\}$. Also, let s^* be the one colored first among all such s^*. Clearly, s^* was colored by the second player, since the first player uses the selection rules. If the coloring of s^* occurs after that of y_2, then the first player would color x when y_2 is colored. Hence s^* is colored before y_2 and after y_1, and the first player's response to this is to color y, again a contradiction. Thus, we have proved that x is the vertex the first player would color after the second player colored vertex y_2, a contradiction.

Thus we have shown that U is an independent vertex set whenever $U^* = \emptyset$ and therefore $\chi_g^*(T) \leqslant 3$. □

4. k-INDUCTIVE GRAPHS

Let $G = (V, E)$ be a graph and let L be a linear order on the vertex set V. For each vertex $x \in V$, we define *the back degree of x relative to L* as $|\{y \in V : xy \in E \text{ and } x > y \text{ in } L\}|$. The *back degree of L* is then defined as the maximum back degree of the vertices relative to L. The graph $G = (V, E)$ is said to be *k-inductive* if there is a linear order L on V which has back degree at most k. If G is k-inductive, then $\chi(G) \leqslant k + 1$.

Again, let L be a linear order on the vertex set V of a graph $G = (V, E)$. We define the *arrangeability of x relative to L* as $|\{y \in V : y \leqslant x \text{ in } L \text{ and there is some } z \in V \text{ with } yz \in E,\ xz \in E \text{ and } x < z \text{ in } L\}|$.

The *arrangeability* of L is then the maximum value of the arrangeability of the vertices relative to L. A graph G is *p-arrangeable* if there is a linear order L on the vertices having arrangeability at most P. It is readily seen that a p-arrangeable graph G is p-inductive.

In [5], Faigle, Kern, Kierstead and Trotter showed that the family of bipartite graphs has infinite game chromatic number by proving the following result for a family of 2-inductive graphs.

THEOREM 6. *There is an infinite class of* 2*-inductive graphs G of order n such that* $\chi_g(G) \geqslant \frac{1}{3}\log_2 n$.

In fact, from the examples that they used in their proof, it is not very difficult to see that there is a constant c such that $\chi_g(S(K_n)) \geqslant c \log_2 n$, where $S(K_n)$ is the subdivision of the complete graph K_n of order n. Later, Kierstead and Trotter [6] proved the following very interesting result.

THEOREM 7. *Let* $G = (V, E)$ *be a p-arrangeable graph and let* $\chi(G) = r$. *Then* $\chi_g(G) \leqslant 2rp + 1$.

Considering the game chromatic number II, the following result holds.

THEOREM 8. *Let k be a positive integer and let G be a k-inductive graph of order n. Then,* $\chi_g^*(G) \leqslant c \log n + 1$, *where* $c = 3k/\log\frac{3}{2}$.

PROOF. Let the first and second players alternately color the graph G with $3k$ colors until a new color must be introduced. Let U_1 denote the set of all uncolored vertices and let G_1 denote the subgraph of G induced by U_1. Clearly, $d(v) \geqslant 3k$ for each $v \in U_1$. Since G is a k-inductive graph,

$$\sum_{v \in V(G)} d_G(v) \leqslant 2kn.$$

From above inequality, we see that $|\{v \mid d_G(v) \geqslant 3k\}| \leqslant \frac{2}{3}n$. Thus, $|U_1| \leqslant \frac{2}{3}n$.

The first and second players color the vertices of U_1 with $3k$ new colors until a new color must be introduced. Let U_2 denote the set of all uncolored vertices and let G_2 denote the subgraph of G induced by U_2. Using the same argument as above, we can show that $|U_2| \leq \frac{2}{3}|U_1| \leq (\frac{2}{3})^2 n$. Continue this process; generally, let U_t denote the set of all uncolored vertices after $3kt$ colors have been used and a new color must be introduced in order to keep the graph colored properly. Also, let G_t be the subgraph induced by the vertex set U_t. By induction, we can prove that $|U_t| \leq (\frac{2}{3})^t n$. Since $(\frac{2}{3})^{\log n/\log 1.5} n \leq 1$, all vertices will have been colored when no more than $\log n/\log 1.5 + 1$ colors are used. $\square$

On the other hand, we have the following result.

THEOREM 9. *Let p be a positive integer. There is a 2-inductive graph G_p of order $|V(G_p)| \leq 3^{p(p+1)}$ such that $\chi_g^*(G_p) \geq p$.*

PROOF. The graphs $G_1, G_2, \ldots, G_p$ will be constructed inductively as follows. Initially, let $G_1 = \overline{K_2}$, two independent vertices, and denote the vertex set of G_1 by $V_1(G_1)$.

To construct the graph G_2, let $V_2(G_2)$ be a vertex set of $3^2 = 9$ vertices. For each pair of vertices $u, v \in V_2(G_2)$, let $G_1(u, v)$ be a copy of G_1 and join both vertices u and v to all vertices in $V_1(G_1(u, v))$. It is readily seen that G_2 is the subdivision of a complete multiple edge graph K_9, where every pair of vertices of K_9 has two edges between them.

Suppose that $G_1, G_2, \ldots, G_{p-1}$ have been constructed. Let $V_p(G_p)$ be a set of new vertices with $|V_p(G_p))| = 3^p$. For each pair of vertices $u, v \in V_p(G_p)$ and each $i = 1, \ldots, p-1$, let $G_i(u, v)$ be a copy of G_i and join both vertices u and v to every vertex in $V_i(G_i(u, v))$. Then, we will show that the following claim holds.

CLAIM 1. *Suppose that two players play the game on the vertices of G_p with the condition that if the color i has been used, then both players can use any color from 1 to i. The second player has a strategy to color the vertices such that at any step either there is a vertex in $V_p(G_p)$ that has been colored with the color p or he has a chance to color a vertex in $V_p(G_p)$ with the color p.*

Clearly, claim 1 is true for the graph G_1 Suppose that Claim 1 is true for the graphs $G_1, G_2, \ldots, G_{p-1}$. We will prove that the following claim is true for G_p.

CLAIM 2. *The second player has a strategy to force the condition that at least three vertices in $V_p(G)$ are uncolored if only the colors $1, 2, \ldots, p-1$ are used.*

Notice that $V_p(G_p)$ is an independent vertex set of G_p and every vertex in $V(G_p)$ is adjacent to at most two vertices in $V_p(G_p)$. Hence Claim 1 is true if Claim 2 holds. The strategy for the second player used is described as follows:

(i) If the first player colors a vertex of $G_i(u, v)$ for some $u, v \in V_p(u, v)$ and $i = 1, 2, \ldots, p-1$, then, by the inductive hypothesis, the second player has a strategy to color a vertex in $G_i(u, v)$ such that either there is a vertex in $V_i(G_i)$ which has been colored with the color i or there is a vertex in $V_i(G_i)$ that can be colored with the color i by the second player.

(ii) If the first player colors a vertex $x \in V_p(G_p)$, then the second player chooses, if possible, a pair of vertices u and $v \in V_p(G_p)$ such that neither of them are adjacent to a vertex with the color 1, and colors a vertex in $V_1(G_1(u, v))$ with the color 1. Such a vertex is available from our inductive hypothesis. If there is no such a pair of vertices u

and v, then any vertex $z \in V_p(G_p)$ is either colored by the first player or the vertex z is uncolored and is adjacent to some vertex with the color 1.

Let $V_p^1(G_p)$ be a subset of $V_p(G_p)$ which is uncolored and adjacent to a vertex with the color 1. Notice that every vertex $y \in V_1(G_1(u, v))$ is adjacent to only two vertices in $V_p(G_P)$; thus $|V_p^1(G_p)| \geqslant \frac{1}{3}|V_p(G_p))|$. Then:

(iii) Following step (ii) above, if the first player colors a vertex in $V_p^1(G_1)$, then the second player chooses, if possible, a pair of vertices u and $v \in V_p^1(G_p)$ such that neither of them are adjacent to a vertex with the color 2. If there is no such pair of vertices u and v, let $V_p^2(G_p)$ be a subset of $V_p^1(G_p)$ which is uncolored and adjacent to vertex with the color 2. As above, we have $|V_p^2(G_p)| \geqslant \frac{1}{3}|V_p^1(G_p)| \geqslant (\frac{1}{3})^2 |V_p(G_p)|$. Notice that $N(z)$ contains a vertex with the color 1 and a vertex with the color 2.

(iv) Repeat the above process, then $V_p^1(G_p)$, $V_p^2(G_p), \ldots, V_p^{p-1}(G_p)$ are obtained such that, for each $z \in V_p^i(G_p)$, the neighborhood $N(z)$ contains a vertex with the color j for each $j = 1, 2, \ldots, i$. Furthermore, $|V_p^i(G_p)| \geqslant \frac{1}{3}|V_p^{i-1}(G_p)|$. Hence, $|V_p^{p-1}(G_p)| \geqslant (\frac{1}{3})^{p-1}|V_p(G_p)| = 3$.

It is readily seen from the above that Claim 2 is proven.

Notice that

$$|V(G_p)| = \binom{3^p}{2}[|V(G_1)| + |V(G_2)| + \cdots + |V(G_{p-1})|] + 3^p, \qquad \text{for } p \geqslant 2.$$

Then, $|V(G_p)| \geqslant \sum_{i=1}^{p-1} |V(G_i)| + 1$ for $p \geqslant 2$. Thus,

$$|V(G_p)| \leqslant \binom{3^p}{2}(2|V(G_{p-1})| - 1) + 3^p \leqslant 3^{2p}|V(G_{p-1})|, \qquad \text{for } p \geqslant 3.$$

It is readily seen that the above inequality is also true for $q = 2$. Therefore, $|V(G_p)| \leqslant 3^{p(p+1)}$. □

Let $n = |V(G_p)|$. From the above inequality, we can deduce that $p > c\sqrt{\log n}$. Thus, there are an infinite number of integers n such that there is a 2-inductive graph G_n of order n such that $\chi_g^*(G) \leqslant c\sqrt{\log n}$ for some constant c.

Note that all subdivision graphs are 2-inductive. We will show that $\chi_g^*(G) \leqslant 3$ if G is the subdivision of some graph. In fact, we prove a more general result.

Theorem 10. *Let k be a positive integer and let G be a graph such that the vertex subset $\{v \in V : d(v) > k\}$ forms an independent set of G, then $\chi_g^*(G) \leqslant k + 1$.*

Proof. Let $X = \{1, 2, \ldots, k, k+1\}$ be a color set. Initially, the two players color G using only colors $1, 2, \ldots, k$. This is continued for as long as possible. Then, it is sufficient to show that all uncolored vertices form an independent set. Suppose, to the contrary, there are a pair of uncolored vertices x and y such that $xy \in E$. Then, one of them, say x, must have degree no more than k. Since y is not yet colored, x can be colored properly by one of the colors $1, 2, \ldots, k$, a contradiction.

Corollary 1. *For any graph G, let $S(G)$ denote the subdivision of G. Then, $\chi_g^*(S(G)) \leqslant 3$.*

Acknowledgements

The research of the first author was partially supported under NSA Grant MDA 904-94-H-2060 and ND-EPSCoR # 4963. The research of the second author was partially funded under NSF Grant DMS-9400530.

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Received 12 January 1993 and accepted 18 November 1994

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