ON INDUCED RAMSEY NUMBERS FOR MULTIPLE COPIES OF GRAPHS

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ABSTRACT. We say that a graph F *strongly arrows* a pair of graphs (G, H) and write $F \stackrel{\text{ind}}{\longrightarrow} (G, H)$ if any colouring of its edges with red and blue leads to either a red G or a blue H appearing as induced subgraphs of F. *The induced Ramsey number*, $IR(G, H)$ is defined as $\min\{|V(F)| : F \stackrel{\text{ind}}{\longrightarrow} (G, H)\}$. We consider the connection between the induced Ramsey number for a pair of two connected graphs $IR(G, H)$ and the induced Ramsey number for multiple copies of these graphs $IR(sG, tH)$, where xG denotes the pairwise vertex-disjoint union of x copies of G. It is easy to see that if $F \stackrel{\text{ind}}{\longrightarrow} (G, H)$ then $(s + t - 1)F \stackrel{\text{ind}}{\longrightarrow} (sG, tH)$. This implies that

$$IR(sG, tH) \leq (s + t - 1)IR(G, H).$$

For all known results on induced Ramsey numbers for multiple copies, the inequality above holds as equality. We show that there are infinite classes of graphs for which the inequality above is strict and moreover, $IR(sG, tH)$ could be arbitrarily smaller than $(s + t - 1)IR(G, H)$. On the other hand, we provide further examples of classes of graphs for which the inequality above holds as equality.

1. INTRODUCTION

We say that a graph F *strongly arrows* a pair of graphs (G, H) and write $F \stackrel{\text{ind}}{\longrightarrow} (G, H)$ if any colouring of its edges with red and blue leads to either a red copy of G or a blue copy of H appearing as induced subgraphs of F. We call the graph F *strongly arrowing graph*. Here, a graph G is an *induced subgraph* of a graph F, denoted by $G \prec F$, if G is a subgraph of F and two vertices of G form an edge in G if and only if they form an edge in F. A *copy of a graph* is an isomorphic image of the graph. When it is clear from context, we simply write F instead of a copy of F. For graphs G and H, the *induced Ramsey number*, $IR(G, H)$ is defined as $\min\{|V(F)| : F \stackrel{\text{ind}}{\longrightarrow} (G, H)\}$. It is a generalization of classical Ramsey numbers $R(G, H)$, where we color the edges of a complete graph and do not require the monochromatic

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copies to be induced, i.e., $R(G, H)$ is the smallest integer n such that any edge coloring of K_n, a complete graph on n vertices, in red and blue contains either a red copy of G and a blue copy of H as a subgraph. It is a corollary of the famous theorem of Ramsey that these numbers are always finite.

The existence of the induced Ramsey numbers is not obvious and it was a subject of an intensive study. Finally it was proved independently by Deuber [8], Erdős, Hajnal, and Pósa [12], and Rödl [23, 24]. Since in case of complete graphs the induced subgraph is the same as the subgraph it is obvious that $IR(K_m, K_n) = R(K_m, K_n)$. When at least one of the graphs in the pair is not complete these functions may differ.

The known results on induced Ramsey numbers are mostly of asymptotic type and mostly concern the upper bounds. Erdős conjectured [11] that there is a positive constant c such that every graph G with n vertices satisfies $IR(G, G) \leq 2^{cn}$. The most recent result in that direction is that of Conlon, Fox, and Sudakov [5] who showed that $IR(G, G) \leq 2^{cn \log n}$ improving the earlier result of Kohayakawa, Prömel and Rödl [20] that stated that $IR(G, G) \leq 2^{cn(\log n)^2}$. The known upper bounds are obtained either by probabilistic ([2, 19, 20, 22]) or by constructive methods [25]. A comparision of results of both types can be found in the paper of Shaefer and Shah [25]. Fox and Sudakov in [13] present a unified approach to proving Ramsey-type theorems for graphs with a forbidden induced subgraph which can be used in finding explicit constructions for upper bounds on various induced Ramsey numbers.

Simple lower bounds on induced Ramsey numbers follow from classical Ramsey numbers: $IR(G, H) \geq R(G, H)$. Another general approach for the lower bounds on $IR(G, H)$, where H has chromatic number k is to partition the vertex set of a given graph in $k - 1$ parts such that each part does not induce G, color all edges within the parts red and all edges between the parts blue. The number of such parts is dictated by generalized chromatic numbers, see among others a paper by Albertson, Jamison, Hedetniemi, and Locke [1] and the references therein.

Unfortunately these lower bounds are not strong enough and the best lower bounds are obtained by a careful structural analysis of a given graph. Recently a step in that direction was done by Gorgol in [15]. She showed that the lower bound for the induced Ramsey number for a connected graph G with an independence number α versus a graph H with the clique number ω can be expressed as $(\alpha-1)\frac{\omega(\omega-1)}{2}+\omega$.

We focus on induced Ramsey numbers for multiple copies of connected graphs. The ordinary Ramsey numbers for multiple copies of connected graphs were considered by Burr, Erdős, and Spencer in [4].

Let t be a positive integer and F be a graph. Recall that tF denotes a graph that is a pairwise vertex-disjoint union of t copies of F. Consider graphs G and H such that $F \stackrel{\text{ind}}{\longrightarrow} (G, H)$. Consider $s + t - 1$ vertex disjoint copies of F and color the edges of the resulting graph in red and blue. Then each copy of F will either have a red induced copy of G or a blue induced copy of H. The pigeonhole principle implies that there will be either a red copy of sG or a blue copy of tH. This gives the following.

Observation 1. *Let G, H be graphs. If $F \stackrel{ind}{\longrightarrow} (G, H)$ then*

$$(s + t - 1)F \stackrel{ind}{\longrightarrow} (sG, tH).$$

Thus

$$IR(sG, tH) \leq (s + t - 1)IR(G, H). \tag{1}$$

For all known so far results on induced Ramsey numbers for multiple copies, the inequality above holds as equality. We raised the following question: Does there exist a pair of graphs (G, H) and a graph X such that

$$|V(X)| < (s + t - 1)IR(G, H) \text{ and } X \stackrel{\text{ind}}{\longrightarrow} (sG, tH)?$$

We answer this question in a positive by showing that there are infinite classes of graphs where the inequality above is strict and moreover, $IR(sG, tH)$ is arbitrarily smaller than $(s+t-1)IR(G, H)$. On the other hand, we provide further examples of classes of graphs for which the inequality above holds as equality.

Already in the case when $H = 2K_2$, we observe a different behavior of $IR(G, H)$ depending on G.

Theorem 6. *Let G be a connected graph, s be an integer, $s \geq |V(G)|$. Then $IR(sG, 2K_2) = (s + 1)IR(G, K_2) = (s + 1)|V(G)|$.*

I.e., in this case the inequality (1) holds as an equality. Let P_n denote a path on n vertices.

Theorem 8. (1) *For an integer $n \geq 5$, $IR(P_n, 2K_2) = n + 2 < 2n = 2IR(P_n, K_2)$.*

I.e., in this case the inequality (1) is strict and provides an arbitrarily large gap between $IR(sG, tH)$ and $(s+t-1)IR(G,H)$. We prove these theorems and more results on paths and matchings in Section 3.

We find bounds on induced Ramsey numbers for short paths and complete graphs in Section 5. Finally, we prove the following result in Section 6.

Theorem 12. *Let k be an integer, $k \geq 2$. Then $IR(K_3, kK_3) = 6k$.*

We give known results on induced Ramsey numbers of multiple copies of graphs in Section 2. In the last section we state some general observations.

2. Known results on induced Ramsey numbers for multiple copies

First we introduce some basic notation. For a graph F and subsets of vertices S and S', $F[S]$ denotes a graph induced by S, $F[S,S']$ denotes a bipartite subgraph of F containing all edges with one endpoint in S and another in S', $F[\{x\}, S]$ is denoted $F[x,S]$. We denote the vertex and the edge sets of F by $V(F)$ and $E(F)$, respectively.

For graphs G, H, the vertex-disjoint union of G and H is denoted $G \cup H$, $G \setminus H$ denotes a graph obtained from G by removing $V(H)$. The independence number of a graph G, i.e. the size of the largest set of mutually nonadjacent vertices, is denoted $\alpha(G)$. The symbols P_n, K_n, and S_n stand for a path on n vertices, a complete graph on n vertices, and a star with n edges. For all other graph theoretic notions we refer the reader to the books of West [26] and Diestel [9].

For any n-vertex graph G, $IR(K_2, G) = n$. Gorgol and Łuczak [16] obtained the exact value of induced Ramsey number for a matching versus a complete graph and Grünewald for two matchings [17].

Theorem 1. [16] *Let $s \geq 1$ and $n \geq 2$ be integers. Then $IR(sK_2, K_n) = sn$.*

Theorem 2. [17] *Let $s, t \geq 1$ be integers. Then $IR(sK_2, tK_2) = 2(s+t-1)$.*

Kostochka and Sheikh [21] considered the case when one graph in a pair is a P_3.

Theorem 3. [21] *For any positive integers $n_1, \dots, n_m$,*

$$IR(P_3, \bigcup_{i=1}^{m} K_{n_i}) = \sum_{i=1}^{m} \binom{n_i+1}{2} = \sum_{i=1}^{m} IR(P_3, K_{n_i}).$$

Corollary 1. *For any positive integer* t, $IR(P_3, tK_n) = tIR(P_3, K_n) = t\left(\binom{n}{2} + n\right)$.

Theorem 4. [21] *Let* H_i, $i = 1, 2, \dots, m$ *be a complete multipartite graph. Then* $IR(P_3, \bigcup_{i=1}^{m} H_i) = \sum_{i=1}^{m} IR(P_3, H_i)$.

Corollary 2. *Let* H *be a complete multipartite graph and* t *be a positive integer. Then* $IR(P_3, tH) = tIR(P_3, H)$.

Theorem 5. [21] *For any positive integer* s, $7s \geq IR(P_3, sP_4) \geq 6.1s$.

3. Induced Ramsey numbers G versus $2K_2$

Obviously $IR(G, K_2) = |V(G)|$. We consider $IR(sG, 2K_2)$. The next theorem shows that if s is large enough, the equality in (1) holds.

Theorem 6. *Let* G *be a connected graph and* $s \geq |V(G)|$. *Then* $IR(sG, 2K_2) = (s+1)|V(G)| = (s+1)IR(G, K_2)$.

Proof. The inequality $IR(sG, 2K_2) \leq (s+1)|V(G)|$ follows from (1). We shall show next that $IR(sG, 2K_2) \geq (s+1)|V(G)|$ by proving that any graph on $(s+1)|V(G)| - 1$ vertices can be edge-colored red and blue such that there is neither red induced copy of sG nor blue induced copy of $2K_2$. Let F be an arbitrary graph on $(s+1)|V(G)| - 1$ vertices. We say that an induced subgraph of F isomorphic to sG is a *bundle*. Assume that any red-blue edge coloring of F contains either a red bundle of a blue $2K_2$ as an induced subgraph.

We see that F contains at least one bundle, otherwise we can color all edges of F red. Let $G_1, \dots, G_s$ be copies of G forming a bundle, with respective vertex sets $X_1, \dots, X_s$, let Y be the set of remaining vertices, i.e., $Y = V(F) - V(G_1 \cup \cdots \cup G_s)$. For any other bundle in F, we say that this bundle intersects X_i nontrivially if it does not contain X_i in is vertex set and thus does not contain respective induced copy of G. Note that each bundle contains at least one vertex from each X_i because otherwise the total number of vertices in the bundle is at most $s|G| + s - 1 - |G| < s|G|$. Note that $|Y| = |V(G)| - 1 \leq s - 1$.

For a fixed $i \in \{1, \dots, s\}$, color an edge of G_i blue and color all other edges of F red. Then we see that there is a red bundle H_i. This bundle intersects X_i nontrivially. Thus the bundle contains a copy of G with vertices in X_i and Y. Since G is connected, there is an edge between X_i and Y for each $i = 1, \dots, s$.

Let Q be an auxiliary bipartite graph with one part Y and another $X = \{X_1, \ldots, X_s\}$ and $X_i y \in E(Q)$ iff there is a bundle containing a copy of G with an edge between y and X_i. By the previous remark $|N_Q(X_i)| \geq 1$. Consider a smallest subset X' of X such that $|N_Q(X')| < |X'|$. Note that X' is well defined since $N_Q(X) \subseteq Y$ and $|Y| < |X|$. We see that $|X'| > 1$.

Consider a bundle H intersecting the largest number t of X_i's nontrivially, for $X_i \in X'$. Let $X' = X'' \cup X'''$, where for each $X_i \in X''$, H intersects X_i nontrivially, and for each $X_i \in X'''$, $X_i \subseteq V(H)$, i.e. $G_i \subseteq H$. Let H'' be a union of copies of G from H that intersect members of X''. Observe that $H'' = t'G$ for $t' \geq t = |X''|$. Indeed, otherwise

$$|\cup_{X_i \in X''} X_i - V(H)| \geq t|G| - t'(|G| - 1) \geq t|G| - (t-1)|G| \geq |G|.$$

Thus the number of vertices of F not in H is at least $|G|$, a contradiction. Since each copy of G in H'' has a vertex in Y, we see that $|V(H'') \cap Y| \geq |X''|$. Consider X'''. Since there are no edges of F between $\cup_{X_i \in X'''} X_i$ and $V(H) \cap Y$ and there are no edges of Q between X' and $Y - N(X')$, $N_Q(X''') \subseteq N(X') - (V(H'') \cap Y)$. Thus $|N_Q(X''')| \leq |N(X')| - |X''| < |X'| - |X''| = |X'''|$, a contradiction to minimality of X'. □

In the next theorem we show the lower bound on the induced Ramsey number for a pair $(G, 2K_2)$.

Theorem 7. *Let G be a graph without isolated vertices. Then*

$$IR(G, 2K_2) \geq |V(G)| + 2.$$

Proof. Let G have n vertices. Consider an arbitrary graph F on $n+1$ vertices. We shall show that F can be edge-colored so that there is no induced red G and no induced blue $2K_2$ in F. If there is no induced G in F, color all edges of F red. So assume there is an induced copy G' of G. Consider the vertex v of F not in G'. Let u be a vertex of G' incident to v if such exists, let u be an arbitrary vertex of G' otherwise. Color all edges incident to v blue and color one edge of G' incident to u blue; color all other edges red. This is a desired coloring, so $IR(G, 2K_2) > n+1$. □

Corollary 3. *Let G be any graph on n vertices and no isolated vertices. Then $n+2 \leq IR(G, 2K_2) \leq 2n$. Moreover both bounds can be attained.*

Proof. Theorem 7 gives the lower bound and Observation 1 implies $IR(G, 2K_2) \leq 2IR(G, K_2) = 2n$, giving an upper bound. Since

$IR(K_2, 2K_2) = 4$, both bounds are tight. We can see that the bounds are tight for some graphs G with arbitrarily many vertices since $IR(P_n, 2K_2) = n + 2$, for $n \geq 5$, as we shall see in the next section and $IR(K_n, 2K_2) = 2n$ by [16]. $\square$

In Theorem 8 we show that this lower bound is sharp for instance when G is a path on at least five vertices.

4. Paths and matchings

Let for a, b positive integers the symbol $\mathrm{rem}(a, b)$ denotes the reminder of a division a by b.

Theorem 8. *Let n, s, t be positive integers. Then*

(1) $IR(P_n, 2K_2) = \begin{cases} n+3, & n = 3, 4, \\ n+2, & n \geq 5, \end{cases}$
(2) $IR(sP_n, 2K_2) \leq sn + s + 1,\ 2 \leq s \leq n-1,\ n \geq 4,$
(3) $IR(sP_n, 2K_2) = sn + s + 1,\ s = 2, 3,\ n \geq 4,$
(4) $IR(sP_n, 2K_2) = (s + 1)n,\ s \geq n,\ n \geq 4,$
(5) $IR(sP_3, 2K_2) = 3s + 3 = (s + 1)IR(P_3, K_2),\ s \geq 1.$
(6) $IR(P_3, tK_2) = 3t,\ t \geq 1,$
(7) $3t + 1 \leq IR(P_4, tK_2) \leq 7\lfloor t/2 \rfloor + 4 \cdot \mathrm{rem}(t, 2),\ t \geq 1,$
(8) $IR(P_n, tK_2) \leq \lceil t/2 \rceil n + t - \mathrm{rem}(t, 2),\ n \geq 5,\ t \geq 1,$
(9) $IR(P_3, tP_3) = 4t,\ t \geq 1.$

Proof. (1) To show that $IR(P_n, 2K_2) \leq n + 2$ for $n \geq 5$, we shall prove that $C_{n+2} \xrightarrow{\text{ind}} (P_n, 2K_2)$. Consider an edge-coloring of C in red and blue. If there is a red P_n, we are done. Assume that there are no red P_n's. Since deleting any two consecutive vertices in C leaves P_n and it is not red, we see using that fact that $n + 2 \geq 7$ that there are two blue edges at distance at least two in C. These edges form induced $2K_2$. Thus $IR(P_n, 2K_2) \leq n+2$. The lower bound comes directly from Theorem 7.

We know that $IR(P_3, 2K_2) = 6$ (cf. Corollary 1). In turn for $(P_4, 2K_2)$ the argument above shows that $C_7 \xrightarrow{\text{ind}} (P_4, 2K_2)$. Consider a graph F on 6 vertices. We shall show that it can be colored with no red induced P_4 and no blue induced $2K_2$. We can assume that there is an induced copy P of P_4 in F, otherwise we can color all edges red. Let $P = (x_1, x_2, x_3, x_4)$ and let x, y be the vertices of F not in P.

If $xy \in E(F)$, color xy, x_1x_2, and x_3x_4 red and all remaining edges blue. Then there is no red P_3 and any blue edge is incident to x_2 or x_3 that are adjacent, thus there is no induced blue $2K_2$.

If $xy \notin E(F)$ and x, y, x_1, x_4 induce a P_4, say with x_4 being an endpoint of this P_4, color all edges incident to x_3 and all edges incident to x_4 blue, and the remaining edges red. Then the red edges form a star, thus there is no red P_4, each of the blue edges is incident to one of two adjacent vertices, thus there is no blue induced $2K_2$.

If $xy \notin E(F)$ and x, y, x_1, x_4 does not induce a P_4, color all edges incident to x_2 and all edges incident to x_3 blue, and all other edges red. As before we see that there is no induced blue $2K_2$. The red edges are spanned by x, y, x_1, x_4, that does not induce a P_4.

Thus $IR(P_4, 2K_2) > 6$. Together with the upper bound, we get $IR(P_4, 2K_2) = 7$.

(2) Let $2 \leq s \leq n-1$. Note that $sP_n \prec P_{sn+s-1}$. To see this, delete every $(n+1)^{\text{st}}$ vertex from P_{sn+s-1}. Thus $IR(sP_n, 2K_2) \leq IR(P_{sn+s-1}, 2K_2)$. By item (1) we have that $IR(P_{sn+s-1}, 2K_2) \leq sn + s + 1$. Therefore $IR(sP_n, 2K_2) \leq sn + s + 1$.

(3) The upper bound follows directly from item (2).

As for the lower bound consider a graph F on $ns + s$ vertices. It contains a bundle B (induced copy of sP_n). Let Y be the set of all vertices of F not in B, i.e., $|Y| = s$.

Fix any component P in B and color any three consecutive edges on it blue and all the remaining edges of F red. Then we see that there must be a red bundle, otherwise we are done. Let this bundle be B_1. We see that B_1 can use at most $n-2$ vertices from P, so it uses at least two vertices of Y.

Assume that there is exactly one vertex of Y adjacent to P. If $|V(P) - V(B_1)| = 3$, then $s = 3$ and Y is contained in $V(B_1)$. Moreover, two other paths of B are components of B_1, so they have no neighbors in Y, a contradiction. If $|V(P) - V(B_1)| = 2$, then the remaining $n-2$ vertices of P together with the only one vertex from Y can induce a path on at most $n-1$ vertices, so it could not be a component of B_1.

So we can assume that there are at least two vertices in Y sending edges to P. If B_1 uses at most $n-1$ vertices from each of the remaining $s-1$ components of B, we see that B_1 omits at least $2+(s-1) = s+1$ vertices of F, i.e., it contains at most $ns-1$ vertices, a contradiction. Thus, there is a component P' of B so that all its vertices are contained in B_1. Since B_1 is an induced subgraph of F, P' is a component of B_1 as well. Since P could be chosen to be an arbitrary component of B, let $P = P'$. We see that on one hand P' sends edges to at least two vertices of Y, on the other hand, it does not send edges to $Y \cap B_1$. Since $|Y \cap B_1| \geq 2$, there are at least two vertices of Y that send edges to P' and at least two vertices of Y that do not send edges to P'. So $|Y| \geq 4$, a contradiction to the fact that $|Y| = s \in \{2, 3\}$.

(4) By Theorem 6 $IR(sP_n, 2K_2) = (s+1)n$ for $s \geq n$.

(5) The upper bound $IR(sP_3, 2K_2) \leq 3s+3$ follows from Observation 1. For the lower bound, consider a graph F on $3s+2$ vertices. We shall show that F can be edge-colored so that there is no induced red sP_3 and no induced blue $2K_2$. We can assume that $sP_3 \prec F$ otherwise we can color all edges of F red. Let a_i, b_i, c_i be the vertices of i-th path P_3 and x and y be the remaining vertices of F.

Assume first that for some $i \in [s]$, there is an edge between $\{a_i, c_i\}$ and $\{x, y\}$, assume w.l.o.g., that $a_1x \in E(F)$. Color all edges incident to a_1 and b_1 blue, the rest of the edges red. Then there is no blue $2K_2$ and we must have a red copy B of sP_3. We see that B could contain at most 2 vertices from $\{x, a_1, b_1, c_1\}$ otherwise it would induce a blue edge. Since $|V(B)| = |V(F)| - 2$, B contains exactly two vertices from $\{x, a_1, b_1, c_1\}$ and thus contains y and all paths (a_i, b_i, c_i), $i = 2, \ldots, s$. Thus there is a red P_3 induced by $\{x, y, a_1, b_1, c_1\}$ and containing y. Since all edges incident to a_1 and b_1 are blue, it must be induced by $\{x, y, c_1\}$. Then we see that $F[\{x, y, a_1, b_1, c_1\}]$ contains a C_4 with a pendant edge or a C_5 and thus does not induce $2K_2$. Color $F[\{x, y, a_1, b_1, c_1\}]$ blue and the rest red, it results in a desired coloring.

Now, we can assume that $\{x, y\}$ sends edges only to vertices in $\{b_1, \ldots, b_s\}$. Assume w.l.o.g., that $b_1x \in E(F)$. Color all edges incident to x and to b_1 blue, the rest red. This is a desired coloring.

(6) According to Corollary 1 $IR(P_3, tK_2) = 3t$.

(7) By item (1) we have that $C_7 \xrightarrow{\text{ind}} (P_4, 2K_2)$.

Hence $(t/2)C_7 \xrightarrow{\text{ind}} (P_4, tK_2)$ for t even and $\lfloor t/2 \rfloor C_7 \cup P_4 \xrightarrow{\text{ind}} (P_4, tK_2)$ for t odd. This gives the upper bound.

As for the lower bound let F be an arbitrary graph on $3t$ vertices. We shall prove that there is an edge coloring of F in two colors with no red induced P_4 and no blue induced tK_2. First observe that $tK_2 \prec F$ otherwise we could color all edges of F blue. Let x_iy_i, $i = 1, 2, \ldots, t$, be the edges of this induced matching and z_i, $i = 1, 2, \ldots, t$ be the remaining vertices of F. Color all edges x_iy_i red. We can assume that there is another induced blue tK_2 otherwise we color the remaining edges blue and obtain the desired coloring. As we can take exactly one vertex from each x_iy_i to construct this new matching M, vertices z_i form an independent set and all are involved in M. Without loss of generality we can assume that x_iz_i, $i = 1, 2, \dots, t$, are the edges of M. We color all of them red. So again we can assume that there is one more induced matching $M_1 = tK_2$. The only possibility is that each z_i, $i = 1, 2, \dots, t$ has exactly one neighbor in $\{y_1, y_2, \dots, y_t\}$. Therefore F consists of induced cycles of length divisible by 3, i.e. $F = \bigcup_k C_{3t_k}$. We shall show that induced $C = C_{3\tau}$ can be colored without red induced P_3 and blue induced τK_2. If $\tau = 1$, color $C = C_3$ red. If $\tau > 1$, color $C = C_{3\tau}$ so that the red subgraph forms matching on all but at most one vertex of C. Then there is no red P_3 and the blue subgraph forms a disjoint union of edges and perhaps one P_3. Since consecutive blue edges on C do not form an induced $2K_2$, the largest induced blue matching has at most $\lfloor \frac{3\tau+1}{4} \rfloor < \tau$ edges.

Let M_k be the largest induced blue matching in C_{3t_k}. Since $F = \bigcup_k C_{3t_k}$, the largest induced blue matching in F has the cardinality $\sum_k |M_k| < \sum_k t_k = t$, which completes the proof of the lower bound.

(8) From item (1) we have that $C_{n+2} \xrightarrow{\text{ind}} (P_n, 2K_2)$, $n \geq 5$. Hence $(t/2)C_{n+2} \xrightarrow{\text{ind}} (P_n, tK_2)$ for t even and $\lfloor t/2 \rfloor C_{n+2} \cup P_n \xrightarrow{\text{ind}} (P_n, tK_2)$ for t odd. This gives the upper bound.

(9) This follows immediately from Theorem 4 since $P_3 = K_{2,1}$. □

5. Short paths and complete graphs

As we mentioned Kostochka and Sheikh showed that $IR(P_3, sK_n) = sIR(P_3, K_n) = s\binom{n}{2} + sn$. We consider the case when there are multiple copies of P_3 and one copy of K_n instead.

Theorem 9. *Let $s \geq 1$ and $n \geq 3$. Then*

$$\binom{n+1}{2} + (2s-2)(n-1) \leq IR(sP_3, K_n) \leq sIR(P_3, K_n) = s\binom{n}{2} + sn.$$

Proof. The upper bound follows from Observation 1.

For the lower bound, consider a graph F on the smallest number of vertices such that $F \xrightarrow{\text{ind}} (sP_3, K_n)$. We see that there is a copy of K_n in F, otherwise we can color all edges of F blue.

Let us denote this clique K^0 and colour it red. We see that $F \setminus V(K^0)$ contains a clique K_{n-1} otherwise color K^0 red and the remaining edges of F blue. Denote this clique K^1 and colour $F_1 = F[V(K^0) \cup V(K^1)]$

red. For $s \geq 2$, F_1 does not contain an induced copy of sP_3, so there is no red sP_3. Similarly $F \backslash V(F_1)$ contains a clique K_{n-1} which we denote K^2 . Repeating the above consideration we conclude that apart from K^0 the graph F contains $2s-1$ pairwise vertex disjoint cliques K_{n-1} denoted by K^1, K^2, $\ldots$, K^{2s-2}. Let $F' = F[V(\bigcup_{i=0}^{2s-2} K^i)]$. Color all edges of F' red and color the edges of K^{2s-1} red. Color the remaining edges blue. We see that there is no red sP_3, so there must be a blue K_n. Thus there is a copy of K_{n-2} induced by the vertices of F not in $\bigcup_{i=0}^{2s-1} K^i$. Similarly, we observe that F contains a vertex-disjoint union of F' and a graph K that is a vertex disjoint union of copies of $K_{n-1}, K_{n-2}, \ldots, K_2$. If $V(F) = V(F') \cup V(K)$, we color all edges F' red, all edges of K red and the remaining vertices blue. Note that the blue color class forms an $(n-1)$-partite graph and thus does not contain K_n. So, $F \overset{\text{ind}}{\nrightarrow} (sP_3, K_n)$, a contradiction. Therefore, $|V(F)| > |V(F')| + |V(K)| = n + (2s-2)(n-1) + (n-1) + (n-2) + \ldots + 2$. In particular $|F| \geq \binom{n+1}{2} + (2s-2)(n-1)$. $\square$

A similar argument works for (sG, K_n) with G being a triangle free graph.

We can improve the lower bound on $IR(2P_3, K_3)$ from 10 as given in Theorem 9 to 11.

Theorem 10. *Then* $IR(2P_3, K_3) \ \geq \ 11.$

Proof. Let F be an arbitrary graph on 10 vertices. We shall prove that there is an edge-coloring of F with no red copy of $2P_3$ and no blue copy of K_3, i.e. that $F \overset{\text{ind}}{\nrightarrow} (2P_3, K_3)$.

We can assume that F contains a vertex disjoint union of K_3 and $2K_2$. Indeed, K_3 exists otherwise we can color all edges of F blue and $F \overset{\text{ind}}{\nrightarrow} (2P_3, K_3)$. Coloring the edges of a copy K of K_3 red and all others blue implies that there must be a blue K_3, so there must be a copy K' of K_2 vertex-disjoint from K. Finally, coloring the subgraph of F induced by vertices of K and K' red, and other edges blue, shows that there is a blue K_3, i.e., in particular a K_2 vertex disjoint from $K \cup K'$. So, we indeed can assume that F contains a vertex-disjoint union of K_3 and $2K_2$. Note that any graph containing $K_3 \cup 2K_2$ as a spanning subgraph does not contain $2P_3$ as an induced subgraph.

Case 1. For some copy K'' of a vertex-disjoint union of K_3 and $2K_2$, $F - V(K'')$ is not isomorphic to P_3. In this case, color the edges of K'' and $F - V(K'')$ red and the remaining edges blue. This results in no induced red $2P_3$ and no blue K_3, so $F \overset{\text{ind}}{\not\to} (2P_3, K_3)$.

Case 2. For any copy K'' of a vertex-disjoint union of K_3 and $2K_2$, $F - V(K'')$ is isomorphic to P_3. We have then that F contains a spanning subgraph that is a union of K, K', and P, where P is a copy of an induced P_3, K is isomorphic to K_3, and K' is isomorphic to $2K_2$. By taking an edge e of K', an edge e' of P, we see that the vertices of $F - V(K)$ not incident to e or e' induce a copy of P_3. Thus $F - V(K)$ contains a spanning subgraph that is a union of three copies of P_3 that share exactly one vertex that is an endpoint in each of these P_3's. Then we see that $F - V(K)$ does not contain a copy of an induced $2P_3$. Color all edges of K and all edges of $F - V(K)$ red and the remaining edges blue. There is no induced red $2P_3$ and no blue K_3, so $F \overset{\text{ind}}{\not\to} (2P_3, K_3)$. □

6. Triangles

Ramsey numbers for multiple copies of graphs were considered by Burr, Erdős and Spencer in [4]. Their paper contains, among others, the following result.

Theorem 11. [4] *Let $t \geq s \geq 1$ and $t \geq 2$ be integers. Then $R(sK_3, tK_3) = 2s + 3t$.*

We prove the following.

Theorem 12. *Let t be a positive integer. Then $IR(K_3, tK_3) = 6t$.*

Proof. The upper bound follows immediately from (1):

$$IR(K_3, tK_3) \leq tIR(K_3, K_3) = tR(K_3, K_3) = 6t.$$

For the lower bound, we need a statement on induced matchings.

Claim If G is any graph on n vertices, then there is a partition $V(G) = V_1 \cup V_2$ such that any induced matching M in G contains at most $n/3$ edges with both endpoints in V_1 or in V_2, i.e., $|E(M[V_1] \cup M[V_2])| \leq n/3$.

Assume not, consider a partition and an induced matching M such that M has more than $n/3$ edges with both endpoints in one part of the partition. Let a new partition V_1', V_2' be built so that each edge of M has one endpoint in V_1' and another in V_2', the rest of the vertices are assigned to V_1' or V_2' arbitrarily. Then we see that V_i' has an independent set of size greater than $n/3$. Then any matching in V_i' has strictly less than $|V_i'| - n/3$ edges, $i = 1, 2$. So, any induced matching of G contains less than $|V_1'| + |V_2'| - n/3 - n/3 = n/3$ edges with both endpoint in the same part. This concludes the proof of Claim.

Let F be a graph, $F \to (K_3, tK_3)$. We shall show that $|V(F)| \geq 6k$. We can assume $tK_3 \prec F$ otherwise we could color all edges of F blue. Let a_i, b_i, c_i, $i = 1, 2, \dots, t$ be the vertices of these triangles and X be the set of remaining vertices. Let $X = X' \cup X''$ be a partition of X such that any induced matching of $F[X]$ has at most $|X|/3$ edges with both endpoints in X' or in X''. Such a partition exists by Claim. Color a_ib_i, b_ic_i, $F[a_i, X']$, $F[c_i, X'']$, and $F[X', X'']$ red, $i = 1, \dots, t$, and all remaining edges blue. We see that there is no red triangle. Assume that there is a blue induced copy of tK_3, denote it H. Any blue triangle has at most one vertex in $\{a_i, b_i, c_i\}$ for any $i = 1, \dots, t$. Thus $H[X]$ contains a blue induced matching on t vertices. This matching could have its edges only with both endpoints in X' or both endpoints in X'' since all edges between X' and X'' are red. By the way we chose a partition X', X'', there are at most $|X|/3$ such edges. Thus $|X|/3 \geq t$, i.e., $|X| \geq 3t$. This implies that $|V(F)| \geq 6t$. $\square$

7. Further observations

While the structure of a graph F such that $F \xrightarrow{\text{ind}} (G, tH)$ and $|V(F)| = IR(G, tH) = tIR(G, H)$ is clear, as it is simply a vertex disjoint union of t copies of F' such that $F' \xrightarrow{\text{ind}} (G, H)$, the structure of such graphs F so that $|V(F)| < tIR(G, H)$ is not so clear. We claim that such a graph must be connected.

Remark 1. *Let G, H be arbitrary connected graphs and t be a positive integer. Let for $i = 1, \dots, t$, $f_i = IR(G, iH)$ and F_i be a graph of order f_i such that $F_i \xrightarrow{ind} (G, iH)$. Assume that $f_t < \min_{\sum t_i = t} \sum f_{t_i}$. Then F_t is connected.*

Proof. Assume to the contrary that F_t consists of $m > 1$ components S_1, S_2, $\dots$, S_m. For $j = 1, \dots, m$, let t_j be the largest integer such that $S_j \xrightarrow{\text{ind}} (G, t_jH)$. Obviously $1 \leq t_j \leq t$. We have that $|S_j| \geq f_{t_j}$, so $f_t \geq f_{t_1} + \cdots + f_{t_m}$. Moreover $F_t \xrightarrow{\text{ind}} (G, (t_1 + \cdots + t_m)H)$. Since F_t is a graph of a smallest order such that $F_t \xrightarrow{\text{ind}} (G, tH)$, we have that $t_1 + \cdots + t_m = t$. But we know that $f_t < f_{t_1} + \cdots + f_{t_m}$ since $t_1 + \cdots t_m = t$. A contradiction. $\square$

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