Ergodic Proofs of VDW Theorem

1 Introduction

Van Der Waerden [5] proved the following combinatorial theorem in a combinatorial way

Theorem 1.1 *For all $c \in \mathsf{N}$, $k \in \mathsf{N}$, any c-coloring of Z will have a monochromatic arithmetic progression of length k.*

Furstenberg [1] later proved it using topological methods. We give a detailed treatment of this proof using as much intuition and as little topology as needed. We follow the approach of [3] who in turn followed the approach of [2].

2 Definitions from Topology

Def 2.1 X is a *metric space* if there exists a function $d : X \times X \rightarrow \mathsf{R}^{\geq 0}$ (called a metric) with the following properties.

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$ (this is called the triangle inequality).

Def 2.2 Let X, Y be metric spaces with metrics d_X and d_Y.

1. If $x \in X$ and $\epsilon > 0$ then $B(x, \epsilon) = \{y \mid d_X(x, y) < \epsilon\}$. Sets of this form are called *balls*.
2. Let $A \subseteq X$ and $x \in X$. x is a *limit point of A* if

$$(\forall \epsilon > 0)(\exists y \in A)[d(x, y) < \epsilon].$$

3. If $x_1, x_2, \ldots \in X$ then $\lim_i x_i = x$ means $(\forall \epsilon > 0)(\exists i)(\forall j)[j \geq i \Rightarrow x_j \in B(x, \epsilon)]$.
4. Let $T : X \rightarrow Y$.
   (a) T is *continuous* if for all $x, x_1, x_2, \ldots \in X$

$$\lim_i x_i = x \Rightarrow \lim_i T(x_i) = T(x).$$

(b) T is *uniformly continuous* if

$$(\forall\epsilon)(\exists\delta)(\forall x, y \in X)[d_X(x,y) < \delta \Rightarrow d_Y(T(x),T(y)) < \epsilon].$$

5. T is *bi-continuous* if T is a bijection, T is continuous, and T^- is continuous.
6. T is *bi-unif-continuous* if T is a bijection, T is uniformly continuous, and T^- is uniformly continuous.
7. If $A \subseteq X$ then
   (a) A' is the set of all limit points of A.
   (b) $\mathrm{cl}(A) = A \cup A'$. (This is called the *closure of* A).
8. A set $A \subseteq X$ is *closed under limit points* if every limit point of A is in A.

Fact 2.3 *If X is a metric space and $A \subseteq X$ then $\mathrm{cl}(A)$ is closed under limit points. That is, if x is a limit point of $\mathrm{cl}(A)$ then $x \in \mathrm{cl}(A)$. Hence $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.*

Note 2.4 The intention in defining the closure of a set A is to obtain the smallest set that contains A that is also closed under limit points. In a general topological space the closure of a set A is the intersection of all closed sets that contain A. Alternatively one can define the closure to be $A \cup A' \cup A'' \cup \cdots$. That $\cdots$ is not quite what is seems- it may need to go into transfinite ordinals (you do not need to know what transfinite ordinals are for this paper). Fortunately we are looking at metric spaces where $\mathrm{cl}(A) = A \cup A'$ suffices. More precisely, our definition agrees with the standard one in a metric space.

Example 2.5

1. $[0,1]$ with $d(x,y) = |x-y|$ (the usual definition of distance).
   (a) If $A = (\frac{1}{2}, \frac{3}{4})$ then $\mathrm{cl}(A) = [\frac{1}{2}, \frac{3}{4}]$.
   (b) If $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ then $\mathrm{cl}(A) = A \cup \{0\}$.
   (c) $\mathrm{cl}(\mathsf{Q}) = \mathsf{R}$.
   (d) Fix $c \in \mathsf{N}$. Let BISEQ be the set of all c-colorings of Z. (It is called BISEQ since it is a bi-sequence of colors. A bi-sequence is a sequence in two directions.) We represent elements of BISEQ by $f : \mathsf{Z} \rightarrow [c]$.

2. Let d : BISEQ × BISEQ → $\mathsf{R}^{\geq 0}$ be defined as follows.

$$d(f,g) = \begin{cases} 0 & \text{if } f = g; \\ \frac{1}{1+i} & \text{if } f \neq g \text{ and } i \text{ is least number s.t. } f(i) \neq g(i) \text{ or } f(-i) \neq g(-i); \end{cases}$$

   One can easily verify that $d(f,g)$ is a metric. We will use this in the future alot so the reader is urged to verify it.

3. The function T is defined by $T(f) = g$ where $g(i) = f(i+1)$. One can easily verify that T is bi-unif-continuous. We will use this in the future alot so the reader is urged to verify it.

Notation 2.6 Let $T : X \to X$ be a bijection. Let $n \in \mathsf{N}$.

1. $T^{(n)}(x) = T(T(\cdots T(x)\cdots))$ means that you apply T to x n times.

2. $T^{(-n)}(x) = T^{-}(T^{-}(\cdots T^{-}(x)\cdots))$ means that you apply T^{-} to x n times.

Def 2.7 If X is a metric space and $T : X \to X$ then

$$\begin{aligned} \text{orbit}(x) = & \ \{T^{(i)}(x) \mid i \in \mathsf{N}\} \\ \text{dorbit}(x) = & \ \{T^{(i)}(x) \mid i \in \mathsf{Z}\} \text{ (dorbit stands for for double-orbit)} \end{aligned}$$

Def 2.8 Let X be a metric space, $T : X \to X$ be a bijection, and $x \in X$.

1. $$\text{CLDOT}(x) = \text{cl}(\{\ldots, T^{(-3)}(x), T^{(-2)}(x), \ldots, T^{(2)}(x), T^{(3)}(x), \ldots)$$

   CLDOT(x) stands for *Closure of Double-Orbit of x*.

2. x is *homogeneous* if

$$(\forall y \in \text{CLDOT}(x))[\text{CLDOT}(x) = \text{CLDOT}(y)].$$

3. X is *limit point compact*[1] if every infinite subset of X has a limit point in X.

Example 2.9 Let BISEQ and T be as in Example 2.5.2. Even though BISEQ is formally the functions from Z to $[c]$ we will use colors as the co-domain.

[1] Munkres [4] is the first one to name this concept "limit point compact"; however, the concept has been around for a long time under a variety of names. Originally, what we call "limit point compact" was just called "compact". Since then the concept we call limit point compact has gone by a number of names: Bolzano-Weierstrass property, Frechet Space are two of them. This short history lesson is from Munkres [4] page 178.

1. Let $f \in$ BISEQ be defined by

$$f(x) = \begin{cases} \text{RED} & \text{if } |x| \text{ is a square;} \\ \text{BLUE} & \text{otherwise.} \end{cases}$$

The set $\{T^{(i)}(f) \mid i \in \mathsf{Z}\}$ has one limit point. It is the function

$$(\forall x \in \mathsf{Z})[g(x) = \text{BLUE}].$$

This is because their are arbitrarily long runs of non-squares. For any M there is an $i \in \mathsf{Z}$ such that $T^{(i)}(f)$ and g agree on $\{-M, \ldots, M\}$. Note that

$$d(T^{(i)}(f), g) \leq \frac{1}{M+1}.$$

Hence

$$\text{CLDOT}(f) = \{T^{(i)}(f) \mid i \in \mathsf{Z}\} \cup \{g\}.$$

2. Let $f \in$ BISEQ be defined by

$$f(x) = \begin{cases} \text{RED} & \text{if } x \geq 0 \text{ and } x \text{ is a square or } x \leq 0 \text{ and } x \text{ is not a square;} \\ \text{BLUE} & \text{otherwise.} \end{cases}$$

The set $\{T^{(i)}(f) \mid i \in \mathsf{Z}\}$ has two limit points. They are

$$(\forall x \in \mathsf{Z})[g(x) = \text{BLUE}]$$

and

$$(\forall x \in \mathsf{Z})[h(x) = \text{RED}].$$

This is because their are arbitrarily long runs of REDs and arbitrarily long runs of BLUEs.

$$\text{CLDOT}(f) = \{T^{(i)}(f) \mid i \in \mathsf{Z}\} \cup \{g, h\}.$$

3. We now construct an example of an f such that the number of limit points of $\{T^{(i)}(f) \mid i \in \mathsf{Z}\}$ is infinite. Let $f_j \in$ BISEQ be defined by

$$f_j(x) = \begin{cases} \text{RED} & \text{if } x \geq 0 \text{ and } x \text{ is a } j\text{th power;} \\ \text{BLUE} & \text{otherwise.} \end{cases}$$

Let $I_k = \{2^k, \ldots, 2^{k+1} - 1\}$. Let $a_1, a_2, a_3, \ldots$ be a list of natural numbers so that every single natural number occurs infinitely often. Let $f \in$ BISEQ be defined as follows.

$$f(x) = \begin{cases} f_j(x) & \text{if } x \geq 1,\ x \in I_k \text{ and } j = a_k; \\ \text{BLUE} & \text{if } x \leq 0. \end{cases}$$

For every j there are arbitrarily long segments of f that agree with some translation of f_j. Hence every point f_j is a limit point of $\{T^{(i)} f \mid i \in \mathsf{Z}\}$.

Example 2.10 We show that BISEQ is limit point compact. Let $A \subseteq$ BISEQ be infinite. Let $f_1, f_2, f_3, \ldots \in A$. We construct $f \in$ BISEQ to be a limit point of $f_1, f_2, \ldots$. Let $a_1, a_2, a_3, \ldots$ be an enumeration of the integers.

$$
\begin{aligned}
I_0 = & \quad \mathsf{N} \\
f(a_1) = & \quad \text{least color in } [c] \text{ that occurs infinitely often in } \{f_i(a_1) \mid i \in I_0\} \\
I_1 = & \quad \{i \mid f_i(a_1) = f(a_1)\}
\end{aligned}
$$

Assume that $f(a_1), I_1, f(a_2), I_2, \ldots, f(a_{n-1}), I_{n-1}$ are all defined and that I_{n-1} is infinite.

$$
\begin{aligned}
f(a_n) = & \quad \text{least color in } [c] \text{ that occurs infinitely often in } \{f_i(a_n) \mid i \in I_{n-1}\} \\
I_n = & \quad \{i \mid (\forall j)[1 \le j \le n \Rightarrow f_i(a_j) = f(a_j)]\}
\end{aligned}
$$

Note that I_n is infinite.

Note 2.11 The argument above that BISEQ is limit point compact is a common technique that is often called a *compactness argument*.

Lemma 2.12 *If X is limit point compact, $Y \subseteq X$, and Y is closed under limit points then Y is limit point compact.*

Proof: Let $A \subseteq Y$ be an infinite set. Since X is limit point compact A has a limit point $x \in X$. Since Y is closed under limit points, $x \in Y$. Hence every infinite subset of Y has a limit point in Y, so Y is limit point compact. ∎

Def 2.13 Let X be a metric space and $T : X \to X$ be continuous. Let $x \in X$.

1. The point x is *recurrent for T* if

$$(\forall \epsilon)(\exists n)[d(T^{(n)}(x), x) < \epsilon].$$

   Intuition: If x is recurrent for T then the orbit of x comes close to x infinitely often. Note that this may be very irregular.

2. Let $\epsilon > 0$, $r \in \mathsf{N}$, and $w \in X$. w is (ϵ, r)*-recurrent for T* if

$$(\exists n \in \mathsf{N})[d(T^{(n)}(w), w) < \epsilon \wedge d(T^{(2n)}(w), w) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(w), w) < \epsilon.]$$

   Intuition: If w is (ϵ, r)-recurrent for T then the orbit of w comes within ϵ of w r times on a regular basis.

Example 2.14

1. If $T(x) = x$ then all points are recurrent (this is trivial).

2. Let $T : \mathsf{R} \to \mathsf{R}$ be defined by $T(x) = -x$. Then, for all $x \in \mathsf{R}$, $T(T(x)) = x$ so all points are recurrent.

3. Let $\alpha \in [0, 1]$. Let $T : [0, 1] \to [0, 1]$ be defined by $T(x) = x + \alpha \pmod 1$.
   (a) If $\alpha = 0$ or $\alpha = 1$ then all points are trivially recurrent.
   (b) If $\alpha \in \mathsf{Q}$, $\alpha = \frac{p}{q}$ then it is easy to show that all points are recurrent for the trivial reason that $T^{(q)}(x) = x + q(\frac{p}{q}) \pmod 1 = x$.
   (c) If $\alpha \notin \mathsf{Q}$ then T is recurrent. This requires a real proof.

3 A Theorem in Topology

Def 3.1 Let X be a metric space and $T : X \to X$ be a bijection. (X, T) is *homogeneous* if, for every $x \in X$,

$$X = \mathrm{CLDOT}(x).$$

Example 3.2

Let $X = [0, 1]$, $d(x, y) = |x - y|$, and $T(x) = x + \alpha \pmod 1$.

1. If $\alpha \in \mathsf{Q}$ then (X, T) is not homogeneous.

2. If $\alpha \notin \mathsf{Q}$ then (X, T) is homogeneous.

3. Let $f, g \in \mathrm{BISEQ}$, so $f : \mathsf{Z} \to \{1, 2\}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \equiv 1 \pmod 2; \\ 2 & \text{if } x \equiv 0 \pmod 2 \end{cases}$$

   and

$$g(x) = 3 - f(x).$$

   Let $T : \mathrm{BISEQ} \to \mathrm{BISEQ}$ be defined by

$$T(h)(x) = h(x + 1).$$

   Let $X = \mathrm{CLDOT}(f)$. Note that

$$X = \{f, g\} = \mathrm{CLDOT}(f) = \mathrm{CLDOT}(g).$$

   Hence (X, T) is homogeneous.

4. All of the examples in Example 2.9 are not homogeneous.

The ultimate goal of this section is to show the following.

Theorem 3.3 *Let X be a metric space and $T : X \to X$ be bi-unif-continuous. Assume (X,T) is homogeneous. Then for every $r \in \mathsf{N}$, for every $\epsilon > 0$, T has an (ϵ, r)-recurrent point.*

Important Convention for the Rest of this Section:

1. X is a metric space.
2. T is bi-unif-continuous.
3. (X,T) is homogeneous.

We show the following by a multiple induction.

1. A_r: $(\forall \epsilon > 0)(\exists x, y \in X, n \in \mathsf{N})$
   $d(T^{(n)}(x), y) < \epsilon \wedge d(T^{(2n)}(x), y) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(x), y) < \epsilon$.

   Intuition: There exists two points x, y such that the orbit of x comes very close to y on a regular basis r times.

2. B_r: $(\forall \epsilon > 0)(\forall z \in X)(\exists x \in X, n \in \mathsf{N})$
   $d(T^{(n)}(x), z) < \epsilon \wedge d(T^{(2n)}(x), z) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(x), z) < \epsilon$.

   Intuition: For any z there is an x such that the orbit of x comes very close to z on a regular basis r times.

3. C_r: $(\forall \epsilon > 0)(\forall z \in X)(\exists x \in X)(\exists n \in \mathsf{N})(\exists \epsilon' > 0)$
   $T^{(n)}(B(x, \epsilon')) \subseteq B(z, \epsilon) \wedge T^{(2n)}(B(x, \epsilon')) \subseteq B(z, \epsilon) \wedge \cdots \wedge T^{(rn)}(B(x, \epsilon'))) \subseteq B(z, \epsilon)$.

   Intuition: For any z there is an x such that the orbit of a small ball around x comes very close to z on a regular basis r times.

4. D_r: $(\forall \epsilon > 0)(\exists w \in X, n \in \mathsf{N})$
   $d(T^{(n)}(w), w) < \epsilon \wedge d(T^{(2n)}(w), w) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(w), w) < \epsilon$.

   Intuition: There is a point w such that the orbit of w comes close to w on a regular basis r times. In other words, for all ϵ, there is a w that is (ϵ, r)-recurrent.

Lemma 3.4 $(\forall \epsilon > 0)(\exists M \in \mathsf{N})(\forall x, y \in X)$

$$\min\{d(x, T^{(-M)}(y)), d(x, T^{(-M+1)}(y)), \ldots, d(x, T^{(M)}(y))\} < \epsilon$$

Proof:
Intuition: Since (X, T) is homogeneous, if $x, y \in X$ then x is close to some point in the double-orbit of y (using T).

Assume, by way of contradiction, that $(\exists \epsilon > 0)(\forall M \in \mathsf{N})(\exists x_M, y_M \in X)$

$$\min\{d(x_M, T^{(-M)}(y_M)), d(x_M, T^{(-M+1)}(y_M)), \ldots, d(x_M, T^{(M)}(y_M))\} \geq \epsilon$$

Let $x = \lim_{M\to\infty} x_M$ and $y = \lim_{M\to\infty} y_M$. Since (X, T) is homogeneous (so it is the closure of a set) and Fact 2.3, $x, y \in X$. Since (X, T) is homogeneous

$$X = \{T^{(i)}(y) \mid i \in \mathsf{Z}\} \cup \{T^{(i)}(y) \mid i \in \mathsf{Z}\}'.$$

Since $x \in X$

$$(\exists^\infty i \in \mathsf{Z})[d(x, T^{(i)}(y)) < \epsilon/4].$$

We don't need the $\exists^\infty$, all we need is to have one such I. Let $I \in \mathsf{Z}$ be such that

$$d(x, T^{(I)}(y)) < \epsilon/4$$

Since $T^{(I)}$ is continuous, $\lim_M y_M = y$, and $\lim_M x_M = x$ there exists $M > |I|$ such that

$$d(T^{(I)}(y), T^{(I)}(y_M)) < \epsilon/4 \wedge d(x_M, x) < \epsilon/4.$$

Hence

$$d(x_M, T^{(I)}(y_M)) \leq d(x_M, x) + d(x, T^{(I)}(y)) + d(T^{(I)}(y), T^{(I)}(y_M)) \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon.$$

Hence $d(x_M, T^{(I)}(y_M)) < \epsilon$. This violates the definition of x_M, y_M. ∎

Note 3.5 The above lemma only used that T is continuous, not that T is bi-unif-continuous.

3.1 $A_r \Rightarrow B_r$

Lemma 3.6 *A_r:* $(\forall \epsilon > 0)(\exists x, y \in X, n \in \mathsf{N})$
$d(T^{(n)}(x), y) < \epsilon \wedge d(T^{(2n)}(x), y) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(x), y) < \epsilon$
$\Rightarrow$
B_r: $(\forall \epsilon > 0)(\forall z \in X)(\exists x \in X, n \in \mathsf{N})$
$d(T^{(n)}(x), z) < \epsilon \wedge d(T^{(2n)}(x), z) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(x), z) < \epsilon$.

Proof:
Intuition: By A_r there is an x, y such that the orbit of x will get close to y regularly. Let $z \in X$. Since (X, T) is homogeneous the orbit of y comes close to z. Hence z is close to $T^{(s)}(y)$ and y is close to $T^{(in)}(x)$, so z is close to $T^{(in+s)}(x) = T^{(in)}(T^{(s)}(x))$. So z is close to $T^{(s)}(x)$ on a regular basis.
Note: The proof merely pins down the intuition. If you understand the intuition you may want to skip the proof.

Let $\epsilon > 0$.

1. Let M be from Lemma 3.4 with parameter $\epsilon/3$.
2. Since T is bi-unif-continuous we have that for $s \in \mathsf{Z}$, $|s| \le M$, $T^{(s)}$ is unif-cont. Hence there exists ϵ' such that

$$(\forall a, b \in X)[d(a, b) < \epsilon' \Rightarrow (\forall s \in \mathsf{Z}, |s| \le M)[d(T^{(s)}(a), T^{(s)}(b)) < \epsilon/3].$$

3. Let $x, y \in X$, $n \in \mathsf{N}$ come from A_r with ϵ' as parameter. Note that

$$d(T^{(in)}(x), y) < \epsilon' \text{ for } 1 \le i \le r.$$

Let $z \in X$. Let y be from item 3 above. By the choice of M there exists s, $|s| \le M$, such that

$$d(T^{(s)}(y), z) < \epsilon/3.$$

Since x, y, n satisfy A_r with ϵ' we have

$$d(T^{(in)}(x), y) < \epsilon' \text{ for } 1 \le i \le r.$$

By the definition of ϵ' we have

$$d(T^{(in+s)}(x), T^{(s)}(y)) < \epsilon/3 \text{ for } 1 \le i \le r.$$

Note that

$$d(T^{(in)}(T^{(s)}(x), z)) \le d(T^{(in)}(T^{(s)}(x)), T^{(s)}(y)) + d(T^{(s)}(y), z) \le \epsilon/3 + \epsilon/3 < \epsilon.$$

∎

3.2 $B_r \Rightarrow C_r$

Lemma 3.7 B_r: $(\forall \epsilon > 0)(\forall z \in X)(\exists x \in X, n \in \mathsf{N})$
$d(T^{(n)}(x), z) < \epsilon \wedge d(T^{(2n)}(x), z) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(x), z) < \epsilon$
$\Rightarrow$
C_r: $(\forall \epsilon > 0)(\forall z \in X)(\exists x \in X, n \in \mathsf{N}, \epsilon' > 0)$
$T^{(n)}B(x, \epsilon') \subseteq B(z, \epsilon) \wedge T^{(2n)}(B(x, \epsilon') \subseteq B(z, \epsilon) \wedge \cdots \wedge T^{(rn)}(B(x, \epsilon') \subseteq B(z, \epsilon)$.

Proof:
Intuition: Since the orbit of x is close to z on a regular basis, balls around the orbits of x should also be close to z on the same regular basis.

Let $\epsilon > 0$ and $z \in X$ be given. Use B_r with $\epsilon/3$ to obtain the following:

$$(\exists x \in X, n \in \mathsf{N})[d(T^{(n)}(x), z) < \epsilon/3 \wedge d(T^{(2n)}(x), z) < \epsilon/3 \wedge \cdots \wedge d(T^{(rn)}(x), z) < \epsilon/3].$$

By uniform continuity of $T^{(in)}$ for $1 \le i \le r$ we obtain ϵ' such that

$$(\forall a, b \in X)[d(a, b) < \epsilon' \Rightarrow (\forall i \le r)[d(T^{(in)}(a), T^{(in)}(b)) < \epsilon^2]$$

We use these values of x and ϵ'.
Let $w \in T^{(in)}(B(x, \epsilon'))$. We show that $w \in B(z, \epsilon)$ by showing $d(w, z) < \epsilon$.
Since $w \in T^{(in)}(B(x, \epsilon'))$ we have $w = T^{(in)}(w')$ for $w' \in B(x, \epsilon')$. Since

$$d(x, w') < \epsilon'$$

we have, by the definition of ϵ',

$$d(T^{(in)}(x), T^{(in)}(w')) < \epsilon/3.$$

$$d(z, w) = d(z, T^{(in)}(w')) \le d(z, T^{(in)}(x)) + d(T^{(in)}(x), T^{(in)}(w')) \le \epsilon/3 + \epsilon/3 < \epsilon.$$

Hence $w \in B(z\epsilon)$. ∎

Note 3.8 The above proof used only that T is unif-continuous, not bi-unif-continuous. In fact, the proof does not use that T is a bijection.

3.3 $C_r \Rightarrow D_r$

Lemma 3.9 C_r: $(\forall\epsilon > 0)(\forall z \in X)(\exists x \in X, n \in \mathsf{N}, \epsilon' > 0)$
$T^{(n)}B(x,\epsilon') \subseteq B(z,\epsilon) \wedge T^{(2n)}(B(x,\epsilon') \subseteq B(z,\epsilon) \wedge \cdots \wedge T^{(rn)}(B(x,\epsilon') \subseteq B(z,\epsilon)$
$\Rightarrow$
D_r: $(\forall\epsilon > 0)(\exists w \in X, n \in \mathsf{N})$
$d(T^{(n)}(w),w) < \epsilon \wedge d(T^{(2n)}(w),w) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(w),y) < \epsilon$.

Proof:
Intuition: We use the premise iteratively. Start with a point z_0. Some z_1 has a ball around its orbit close to z_0. Some z_2 has a ball around its orbit close to z_1. Etc. Finally there will be two z_i's that are close: in fact the a ball around the orbit of one is close to the other. This will show the conclusion.

Let $z_0 \in X$. Apply C_r with $\epsilon_0 = \epsilon/2$ and z_0 to obtain z_1, ϵ_1, n_1 such that

$$T^{(in_1)}(B(z_1,\epsilon_1)) \subseteq B(z_0,\epsilon_0) \text{ for } 1 \le i \le r.$$

Apply C_r with ϵ_1 and z_1 to obtain z_2, ϵ_2, n_2 such that

$$T^{(in_2)}(B(z_2,\epsilon_2)) \subseteq B(z_1,\epsilon_1) \text{ for } 1 \le i \le r.$$

Apply C_r with ϵ_2 and z_2 to obtain z_3, ϵ_3, n_3 such that

$$T^{(in_3)}(B(z_3,\epsilon_3)) \subseteq B(z_2,\epsilon_2) \text{ for } 1 \le i \le r.$$

Keep doing this to obtain $z_0, z_1, z_2, \ldots$.
One can easily show that, for all $t < s$, for all i $1 \le i \le r$,

$$T^{(i(n_s+n_{s+1}+\cdots+n_{s+t}))}(B(z_s,\epsilon_s)) \subseteq B(z_t,\epsilon_t)$$

Since X is closed $z_0, z_1, \ldots$ has a limit point. Hence

$$d(z_s,z_t) < \epsilon_0.$$

Using these s,t and letting $n_s + \cdots + n_{s+t} = n$ we obtain

$$T^{(in)}(B(z_s,\epsilon_s)) \subseteq B(z_t,\epsilon_t)$$

Hence

$$d(T^{(in)}(z_s),z_t) < \epsilon_t.$$

Let $w = z_s$. Hence, for $1 \le i \le r$

$$d(T^{(in)}(w),w) \le d(T^{(in)}(z_s),z_s) \le d(T^{(in)}(z_s),z_t) + d(z_t,z_s) < \epsilon_t + \epsilon_0 < \epsilon.$$

∎

3.4 $D_r \Rightarrow A_{r+1}$

Lemma 3.10 *D_r: $(\forall \epsilon > 0)(\exists w \in X, n \in \mathsf{N})$*
$d(T^{(n)}(w), w) < \epsilon \wedge d(T^{(2n)}(w), w) < \epsilon \wedge \cdots \wedge d(T^{(rn)}(w), y) < \epsilon.$
$\Rightarrow$
A_{r+1}: $(\forall \epsilon > 0)(\exists x, y \in X, n \in \mathsf{N})$
$d(T^{(n)}(x), y) < \epsilon \wedge d(T^{(2n)}(x), y) < \epsilon \wedge \ , \ldots , \ d(T^{((r+1)n)}(x), y) < \epsilon.$

Proof:

By D_r and $(\forall x)[d(x,x) = 0]$ we have that there exists a $w \in X$ and $n \in \mathsf{N}$ such that the following hold.

$$\begin{aligned} d(w,w) &< \epsilon \\ d(T^{(n)}(w), w) &< \epsilon \\ d(T^{(2n)}(w), w) &< \epsilon \\ &\vdots \\ d(T^{(rn)}(w), w) &< \epsilon \end{aligned}$$

We rewrite the above equations.

$$\begin{aligned} d(T^{(n)}(T^{(-n)}(w)), w) &< \epsilon \\ d(T^{(2n)}(T^{(-n)}(w)), w) &< \epsilon \\ d(T^{(3n)}(T^{(-n)}(w)), w) &< \epsilon \\ &\vdots \\ d(T^{(rn)}(T^{(-n)}(w)), w) &< \epsilon \\ d(T^{((r+1)n)}(T^{(-n)}(w)), w) &< \epsilon \end{aligned}$$

Let $x = T^{(-n)}(w)$ and $y = w$ to obtain

$$\begin{aligned} d(T^{(n)}(x), y) &< \epsilon \\ d(T^{(2n)}(x), y) &< \epsilon \\ d(T^{(3n)}(x), y) &< \epsilon \\ &\vdots \\ d(T^{(rn)}(x), y) &< \epsilon \\ d(T^{((r+1)n)}(x), y) &< \epsilon \end{aligned}$$

▮

Theorem 3.11 *Assume that*

1. *X is a metric space,*
2. *T is bi-unif-continuous.*
3. *(X, T) is homogeneous.*

For every $r \in \mathsf{N}$, $\epsilon > 0$, *there exists* $w \in X$, $n \in \mathsf{N}$ *such that* w *is* (ϵ, r)*-recurrent.*

Proof:

Recall that A_1 states

$$(\forall \epsilon)(\exists x, y \in X)(\exists n)[d(T^{(n)}(x), y) < \epsilon].$$

Let $x \in X$ be arbitrary and $y = T(y)$. Note that

$$d(T^{(1)}(x), y) = d(T(x), T(x)) = 0 < \epsilon.$$

Hence A_1 is satisfied.

By Lemmas 3.6, 3.7, 3.9, and 3.10 we have $(\forall r \in \mathsf{N})[D_r]$. This is the conclusion we seek. ▮

4 Another Theorem in Topology

Recall the following well known theorem, called **Zorn's Lemma**.

Lemma 4.1 *Let* $(X, \preceq)$ *be a partial order. If every chain has an upper bound then there exists a maximal element.*

Proof: See Appendix TO BE WRITTEN ▮

Lemma 4.2 *Let* X *be a metric space,* $T : X \to X$ *be bi-continuous, and* $x \in X$*. If* $y \in \mathrm{CLDOT}(x)$ *then* $\mathrm{CLDOT}(y) \subseteq \mathrm{CLDOT}(x)$*.*

Proof: Let $y \in \mathrm{CLDOT}(x)$. Then there exists $i_1, i_2, i_3, \ldots \in \mathsf{Z}$ such that

$$T^{(i_1)}(x), T^{(i_2)}(x), T^{(i_3)}(x), \ldots \to y.$$

Let $j \in Z$. Since $T^{(j)}$ is continues

$$T^{(i_1+j)}(x), T^{(i_2+j)}(x), T^{(i_3+j)}(x), \ldots \to T^{(j)}y.$$

Hence, for all $j \in \mathsf{Z}$,

$$T^{(j)}(y) \in \mathrm{cl}\{T^{(i_k+j)}(x) \mid k \in \mathsf{N}\} \subseteq \mathrm{cl}\{T^{(i)}(x) \mid i \in \mathsf{Z}\} = \mathrm{CLDOT}(x).$$

Therefore

$$\{T^{(j)}(y) \mid j \in \mathsf{Z}\} \subseteq \mathrm{CLDOT}(x).$$

By taking cl of both sides we obtain

$$\mathrm{CLDOT}(y) \subseteq \mathrm{CLDOT}(x).$$

▮

Theorem 4.3 *Let X be a limit point compact metric space. Let $T : X \to X$ be a bijection. Then there exists a homogeneous point $x \in X$.*

Proof:

We define the following order on X.

$$x \preceq y \text{ iff } \mathrm{CLDOT}(x) \supseteq \mathrm{CLDOT}(y).$$

This is clearly a partial ordering. We show that this ordering satisfies the premise of Zorn's lemma.

Let C be a chain. If C is finite then clearly it has an upper bound. Hence we assume that C is infinite. Since X is limit point compact there exists x, a limit point of C.

Claim 1: For every $y, z \in C$ such that $y \preceq z$, $z \in \mathrm{CLDOT}(y)$.

Proof: Since $y \preceq z$ we have $\mathrm{CLDOT}(z) \subseteq \mathrm{CLDOT}(y)$. Note that

$$z \in \mathrm{CLDOT}(z) \subseteq \mathrm{CLDOT}(y).$$

End of Proof of Claim 1

Claim 2: For every $y \in C$ $x \in \mathrm{CLDOT}(y)$.

Proof: Let $y_1, y_2, y_3, \ldots$ be such that

1. $y = y_1$,
2. $y_1, y_2, y_3, \ldots \in C$,
3. $y_1 \preceq y_2 \preceq y_3 \preceq \cdots$, and
4. $\lim_i y_i = x$.

Since $y \prec y_2 \prec y_3 \prec \cdots$ we have $(\forall i)[\mathrm{CLDOT}(y) \supseteq \mathrm{CLDOT}(y_i)]$. Hence $(\forall i)[y_i \in \mathrm{CLDOT}(y)]$. Since $\lim_i y_i = x$, $(\forall i)[y_i \in \mathrm{CLDOT}(y)]$, and $\mathrm{CLDOT}(y)$ is closed under limit points, $x \in \mathrm{CLDOT}(y)$.

End of Proof of Claim 2

By Zorn's lemma there exists a maximal element under the ordering $\preceq$. Let this element be x.

Claim 3: x is homogeneous.

Proof: Let $y \in \mathrm{CLDOT}(x)$. We show $\mathrm{CLDOT}(y) = \mathrm{CLDOT}(x)$.

Since $y \in \mathrm{CLDOT}(x)$, $\mathrm{CLDOT}(y) \subseteq \mathrm{CLDOT}(x)$ by Lemma 4.2.

Since x is maximal $\mathrm{CLDOT}(x) \subseteq \mathrm{CLDOT}(y)$.

Hence $\mathrm{CLDOT}(x) = \mathrm{CLDOT}(y)$.

End of Proof of Claim 3 ▮

5 VDW

Theorem 5.1 *For all c, for all k, for every c-coloring of* Z *there exists a monochromatic arithmetic sequence of length k.*

Proof:

Let BISEQ and T be as in Example 2.5.2.

Let $f \in$ BISEQ. Let $Y = \text{CLDOT}(f)$. Since BISEQ is limit point compact and Y is closed under limit points, by Lemma 2.12 Y is limit point compact. By Theorem 4.3 there exists $g \in X$ such that CLDOT(g) is homogeneous. Let $X = \text{CLDOT}(g)$. The premise of Theorem 3.11 is satisfied with X and T. Hence we take the following special case.

There exists $h \in X$, $n \in \mathsf{N}$ such that h is $(\frac{1}{4}, k)$-recurrent. Hence there exists n such that

$$d(h, T^{(n)}(h)), d(h, T^{(2n)}(h)), \ldots, d(h, T^{(rn)}(h)) < \frac{1}{4}.$$

Since for all i, $1 \le i \le r$, $d(h, T^{(in)}(h)) < \frac{1}{4} < \frac{1}{2}$ we have that

$$h(0) = h(n) = h(2n) = \cdots = h(kn).$$

Hence h has an AP of length k. We need to show that f has an AP of length k. Let $\epsilon = \frac{1}{2(kn+1)}$. Since $h \in \text{CLDOT}(g)$ there exists $j \in \mathsf{Z}$ such that

$$d(h, T^{(j)}(g)) < \epsilon.$$

Let ϵ' be such that

$$(\forall a, b \in X)[d(a, b) < \epsilon' \Rightarrow d(T^{(j)}(a), T^{(j)}(b)) < \epsilon].$$

Since $g \in \text{CLDOT}(f)$ there exists $i \in \mathsf{Z}$ such that $d(g, T^{(i)}(f)) < \epsilon'$. By the definition of ϵ' we have

$$d(T^{(j)}(g), T^{(i+j)}(f)) < \epsilon.$$

Hence we have

$$d(h, T^{(i+j)}(f)) \le d(h, T^{(j)}(g)) + d(T^{(j)}(g), T^{(i+j)}f) < 2\epsilon \le \frac{1}{kn+1}.$$

Hence we have that h and $T^{(i+j)}(f)$ agree on $\{0, \ldots, kn\}$. In particular
$h(0) = f(i + j)$.
$h(n) = f(i + j + n)$.
$h(2n) = f(i + j + 2n)$.
$\vdots$

$h(kn) = f(i + j + kn)$.

Since

$$h(0) = h(n) = \cdots = h(kn)$$

we have

$$f(i + j) = f(i + j + n) = f(i + j + 2n) = \cdots = f(i + j + kn).$$

Thus f has a monochromatic arithmetic progression of length k.

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