Three Proofs of the Hypergraph Ramsey Theorem

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1 Introduction

The 3-hypergraph Ramsey numbers $R(3,k)$ (which we will define below) were first shown to exist by Ramsey [8]. His bound on them was enormous (formally the Tower function). Erdös-Rado [4] obtained much better bounds, namely $R(3,k) \le 2^{2^{4k-\Omega(\log k)}}$. Recently Conlon-Fox-Sudakov [3] have obtained $R(3,k) \le 2^{2^{2k-\Omega(\log k)}}$ We present all three proofs. Before starting the second and third proofs we will discuss why it improves the prior one.

We also present extensions of all three proofs to the hypergraph case. The first two are known. The extension of Conlon-Fox-Sudakov seems to be new. The bounds obtained from it yield better upper bounds then were previously known.

2 Notation and Ramsey's Theorem

Def 2.1 Let X be a set and $a \in \mathsf{N}$. Then $\binom{X}{a}$ is the set of all subsets of X of size a.

Def 2.2 Let $a, n \in \mathsf{N}$. The *complete a-hypergraph on n vertices*, denoted K_n^a, is the hypergraph with vertex set $V = [n]$ and edge set $E = \binom{[n]}{a}$

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In this paper a *coloring of a graph or hypergraph* always means a coloring of the *edges*.

Def 2.3 Let $G = (V, E)$ be an a-hypergraph, and let COL be a c-coloring of the edges of G. A set of vertices H is *homogeneous for* COL if every edge in $\binom{H}{a}$ is the same color. We will drop the *for* COL when it is understood.

Note 2.4 Assume we are coloring the edges of the complete a-hypergraph. For $a = 2$ a homogeneous set is a complete monochromatic graph. What about $a = 1$? The complete 1-hypergraph on n vertices is just a set of n points. The edges are the 1-sets of vertices, which are just vertices. A homogeneous set is a set of vertices that are all the same color.

Def 2.5 Let $a, c, k \in \mathsf{N}$. Let $R(a, k, c)$ be the least n such that, for all c-colorings of K_n^a there exists a homogeneous set of size k. We denote $R(a, k, 2)$ by $R(a, k)$. We have not shown that $R(a, k, c)$ exists; however, it does.

Convention 2.6 If we state a theorem of the form (say) $R(a, k) \leq 2^{2^{4k}}$ then it means that $R(a, k)$ exists and is less than the bound given.

We state Ramsey's theorem for 1-hypergraphs (which is trivial) and for 2-hypergraphs (just graphs). The 2-hypergraph case, along with the a-hypergraph case, is due to Ramsey [8] (see also [5, 6, 7]). The bound we give on $R(2, k)$ seems to be folklore (see [6]).

Theorem 2.7 *Let* $k \in \mathsf{N}$.

1. $R(1, k) = 2k - 1$.
2. $R(2, k) \leq \binom{2k-2}{k-1} \leq \frac{2^{2k}}{\sqrt{k}} \leq 2^{2k-\Omega(\log k)}$.
3. *For all* n*, for every 2-coloring of* K_n*, there exists a homogenous set* H *of size at least* $\frac{1}{2}\log_2 n + \Omega(1)$. *(This follows from part 1 easily.)*

Note 2.8 Theorem 2.7.2 has an elementary proof. A more sophisticated proof, by David Conlon [2] yields $R(2,k) \le k^{-D\frac{\log k}{\log\log k}}\binom{2k}{k}$, where D is some constant. A simple probabilistic argument shows that $R(2,k) \ge (1+o(1))\frac{1}{e\sqrt{2}})k2^{k/2}$. A more sophisticated argument by Spencer [9] (see [6]), that uses the Lovasz Local Lemma, shows $R(2,k) \ge (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}$.

Note 2.9 Ramsey's theorem generalizes to c colors to yield the following: $R(2,k,c) \le c^{ck+c-1}$.

We state Ramsey's theorem on a-hypergraphs [8] (see also [6, 7]).

Theorem 2.10 *Let $a, k, c \in \mathsf{N}$. For all $k \in \mathsf{N}$, $R(a,k,c)$ exists.*

3 Summary of Results

All of our proofs are for the 2-color case; however, they can all be easily modified for the c-color case. We will include notes about what the results are for c colors.

We present three proofs of the 3-hypergraph Ramsey Theorem (Theorem 2.10 in the case of $k = 3$ and $c = 2$), due to Ramsey [8], Erdös-Rado [4], and Conlon-Fox-Sudakov [3]. Each proof will yield better and better upper bounds on $R(3,k)$. We then use the ideas in these proofs to present three proofs of the a-hypergraph Ramsey Theorem. The first two proofs are due to Ramsey and Erdös-Rado. The third one uses the ideas of Conlon-Fox-Sudakov but seems to be new. Each proof will yield better and better upper bounds on $R(a,k)$.

We will need both the tower function and Knuth's arrow notation to state the results.

Def 3.1 $\mathrm{TOW}_c(a,k)$ is defined as follows

- $\mathrm{TOW}_c(0,k) = k$
- For all $a \ge 1$, $\mathrm{TOW}_c(a,k) = c^{\mathrm{TOW}_c(a-1,k)}$.

Notation 3.2 If we leave out the subscript on TOW then it is assumed to be 2.

Example 3.3

1. $\mathrm{TOW}(1,k) = 2^k$.

2. $\mathrm{TOW}(2,17k) = 2^{2^{17k}}$.

3. $\mathrm{TOW}_5(2,17k) = 5^{5^{17k}}$.

Notation 3.4

$$a \uparrow^n b = \begin{cases} a & \text{if } n = 0, \\ a^b, & \text{if } n = 1, \\ 1, & \text{if } b = 0, \\ a \uparrow^{n-1} (a \uparrow^n (b-1)), & \text{otherwise.} \end{cases}$$

The list below contains both who proved what bounds and the results we will prove in this paper.

1. Ramsey's proof [8] yields:

   (a) $R(3,k) \le \mathrm{TOW}(2k-1,1)$.

   (b) $R(a,k) \le 2 \uparrow^{a-1} (2k-1)$

2. The Erdös-Rado [4] proof yields:

   (a) $R(3,k) \le 2^{2^{4k-\Omega(\log k)}}$.

   (b) $R(a,k) \le \mathrm{TOW}(a-1, 4k-\Omega(\log k))$.

3. The Conlon-Fox-Sudakov [3] proof yields:

   (a) $R(3,k) \le 2^{A\sqrt{k}2^{2k}}$, where $A = \frac{2}{\sqrt{\pi}} \sim 1.128$.

(b) (This result is new though it uses the techniques of Conlon-Fox-Sudakov.)

$$R(a,k) \le \begin{cases} \mathrm{TOW}(a-1, 2k + O(\log k)) & \text{if } a \text{ is odd} \\ \mathrm{TOW}(a-1, 4k - \Omega(\log k)) & \text{if } a \text{ is even} \end{cases}$$

We will need the following notation

Notation 3.5 PHP stands for Pigeon Hole Principle.

4 The Tower Function

We will need the following lemma about the Tower function. We leave the proof to the reader.

FILL IN - CHECK WHAT IS NEEDED OF LEMMA BELOW

Lemma 4.1 *Let* $0 < \epsilon < 1$. *Let* $b, b_1, b_2, L, L_1, L_2) > 0$.

1. *For* $a \ge 0$, *for almost all* k, $b + \mathrm{TOW}(a, Lk) \le \mathrm{TOW}(a, (L+\epsilon)k)$.
2. *For* $a \ge 1$, *for almost all* k, $b\mathrm{TOW}(a, Lk) \le \mathrm{TOW}(a, (L+\epsilon)k)$. *(To prove this you need Part 1.)*
3. *For* $a \ge 2$, *for almost all* k, $\mathrm{TOW}(a, Lk)^b \le \mathrm{TOW}(a, (L+\epsilon)k)$. *(To prove this you need Part 2.)*
4. *For* $a \ge 1$, *for almost all* k $\mathrm{TOW}(a-1, L_1k)\mathrm{TOW}(a, L_2k) \le TOW(a, (L_2+\epsilon)k)$.
5. *For* $a \ge 2$, *for* $L_1 \in \mathsf{N}$ *there exists* L_2 *such that, for all* k,

$$\mathrm{TOW}(a, Lk - L_1 \log k)^a \le \mathrm{TOW}(a, Lk - L_2 \log k).$$

5 Ramsey's Proof

Theorem 5.1 *For all* k $R(3,k) \le \mathrm{TOW}(2k-1,2)$.

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of K_n^3. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{2k-1}.$$

Here is the basic idea: Let $x_1 = 1$. This induces the following coloring of $\binom{\{2,\ldots,n\}}{2}$:

$$COL^*(x,y) = COL(x_1,x,y).$$

By Theorem 2.7 there exists a homogeneous set for COL^* of size $\frac{1}{2}\log_2 n + \Omega(1)$. Keep that homogeneous set and ignore the remaining points. Let x_2 be the least vertex that has been kept. (bigger than x_1). Repeat the process.

We describe the construction formally.

CONSTRUCTION

Let $V_0 = [n]$.

Assume $1 \le i \le 2k-1$ and that V_{i-1}, $x_1, x_2, \ldots, x_{i-1}$, $c_1, \ldots, c_{i-1}$ are all defined. We define x_i, COL^*, V_i, and c_i:

$$\begin{aligned} x_i &= \text{ the least number in } V_{i-1} \\ (\forall \{x,y\} \in \tbinom{V_{i-1}-\{x_i\}}{2})[COL^*(x,y) &= COL(x_i,x,y)] \\ V_i &= \text{ the largest homogeneous set for } COL^* \\ c_i &= \text{ the color of } V_i \end{aligned}$$

KEY: for all $y, z \in V_i$, $COL(x_i, y, z) = c_i$.

END OF CONSTRUCTION

We have vertices

$$x_1, x_2, \ldots, x_{2k-1}$$

and associated colors

$$c_1, c_2, \ldots, c_{2k-1}.$$

There are only two colors, hence, by PHP, there exists $i_1, \ldots, i_k$ such that $i_1 < \cdots < i_k$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

Denote this color by c, and consider the k vertices

$$H = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}.$$

If $x < y < z \in H$ then, by definition, $COL(x, y, z) = c$. Hence H is homogeneous for COL.

We now see how large n must be so that the construction can be carried out. By Theorem 2.7, if k is large, at every iteration V_i gets reduced by a logarithm and then cut in half and then a constant (how big depends on how large k is) is added. Using this it is easy to show that, if k is large enough,

$$|V_j| \geq \frac{1}{2} \log_2^{(j-1)} n.$$

We want to run this iteration $2k - 1$ times Hence we need

$$|V_{2k-1}| \geq \frac{1}{2}\log_2^{(2k-1)} n \geq 1.$$

We can take $n = \mathrm{TOW}(2k-1, 2)$. ▮

Note 5.2 The proof of Theorem 5.1 generalizes to c-colors to yield

$$R(3, k, c) \leq \mathrm{TOW}_c(ck - (c-1), c).$$

We leave this as an exercise for the reader.

We now prove Ramsey's Theorem for a-hypergraphs.

Theorem 5.3 *For all $a \geq 1$, for all $k \geq 1$, $R(a, k) \leq 2 \uparrow^{a-1} (2k-1)$*

FILL IN - HAVE THE THEOREM BE A RECURRENCE IN PART 1.

Proof sketch: This proof is by induction on a.

Base Case: If $a = 1$ then $R(1, k) = 2k - 1 \leq 2 \uparrow^0 (2k-1) = 2k - 1$.

Induction Step: Assume that the theorem is known for $a-1$-hypergraphs. The construction is similar to that given in the proof of Theorem 5.1.

CONSTRUCTION

Let $V_0 = [n]$.

Assume $1 \leq i \leq 2k-1$ and that $V_{i-1}, x_1, x_2, \ldots, x_{i-1}, c_1, \ldots, c_{i-1}$ are all defined. We define x_i, COL^*, V_i, and c_i:

$$\begin{aligned} x_i = &\quad \text{the least number in } V_{i-1} \\ (\forall A \in \tbinom{V_i - \{x_i\}}{a-1})[COL^*(A) = &\quad COL(x_i \cup A)] \\ V_i = &\quad \text{the largest homogeneous set for } COL^* \\ c_i = &\quad \text{the color of } V_i \end{aligned}$$

KEY POINT: For all $1 \le i \le 2k-1$, $(\forall A \in \binom{V_i - \{x_i\}}{a-1})[COL(x_i \cup A) = c_i]$.

END OF CONSTRUCTION

We have vertices

$$x_1, x_2, \ldots, x_{2k-1}$$

and associated colors

$$c_1, c_2, \ldots, c_{2k-1}.$$

There are only two colors, hence, by PHP, there exists $i_1, \ldots, i_k$ such that $i_1 < \cdots < i_k$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

Denote this color by c, and consider the k vertices

$$H = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}.$$

Clearly $(\forall B \in \binom{V_i - \{x_i\}}{a})[COL(B) = c_i]$. Hence H is homogeneous for COL.

We omit the analysis of how big n must be. ∎

Note 5.4 The proof of Theorem 5.3 generalizes to c colors yielding

$$R(a, k, c) \le c \uparrow^{a-1} (ck - (c-1)).$$

We leave this as an exercise for the reader.

6 The Erdös-Rado Proof

Why does Ramsey's proof yield such large upper bounds? Recall that in Ramsey's proof we do the following:

- Color a *node* by using Ramsey's theorem (on graphs). This cuts the number of nodes down by a log (from m to $\Theta(\log m)$). This is done $2k-1$ times.
- After the nodes are colored we use PHP once. This will cut the number of nodes in half.

The key to the large bounds is the number of times we use Ramsey's theorem. The key insight of the proof by Erdös and Rado [4] is that they use PHP many times but Ramsey's theorem only once. In summary they do the following:

- Color an *edge* by using PHP. This cuts the number of nodes in half. This is done $2^{R(k,2)^2}$ times.
- After *all* the edges of a complete graph are colored we use Ramsey's theorem. This will cut the number of nodes down by a log.

We now proceed formally.

Theorem 6.1 *For all k, $R(3,k) \leq 2^{2^{4k-\Omega(\log k)}}$.*

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of K_n^3. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{R(2,k)}.$$

Recall the definition of a homogeneous set for a coloring of singletons from the note following Definition 2.3. We will use it here.

Here is the intuition: Let $x_1 = 1$. Let $x_2 = 2$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be a homogeneous set for COL^* of size at least $\frac{n-2}{2}$. Let $COL^{**}(x_1, x_2)$ be the color of V_1. Let x_3 be the least vertex left (bigger than x_2).

The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[COL_1^*(y) = COL(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[COL_1^*(y) = COL(x_2, x_3, y)]$$

Let V_2 be a homogeneous set for COL_1^* of size $\frac{|V_1|-1}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Restrict COL_2^* to elements of V_2, though still call it COL_2^*. Let V_2 (reuse var name) be a homogeneous set for COL_2^* of size at least $\frac{|V_2|}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Let x_4 be the least element of V_2. Repeat the process.

We describe the construction formally.

CONSTRUCTION

$$\begin{aligned} x_1 = \;& 1 \\ V_1 = \;& [n] - \{x_1\} \end{aligned}$$

Let $2 \le i \le R(2, k)$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^{**} : \binom{\{x_1,\ldots,x_{i-1}\}}{2} \to [2]$ are defined.

$$\begin{aligned} x_i = \;& \text{the least element of } V_{i-1} \\ V_i = \;& V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).} \end{aligned}$$

We define $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, …, $COL^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

For $j = 1$ to $i - 1$

1. $COL^* : V_i \rightarrow [2]$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.
2. Let V_i be redefined as the largest homogeneous set for COL^*. Note that $|V_i|$ is at least half of its previous size. (Recall that COL^* is a coloring of vertices, not edges, and that a homogenous set is one that is all the same color.)
3. $COL^{**}(x_j, x_i)$ is the color of V_i.

KEY: For all $1 \leq a < b \leq i$, for all $y \in V_i$, $COL(x_a, x_b, y) = COL^{**}(x_a, x_b)$.

END OF CONSTRUCTION

We now have

$$X = \{x_1, x_2, \ldots, x_{R(2,k)}\}.$$

Note that COL^{**} is a 2-coloring of pairs of elements of X. We apply Ramsey's theorem (on graphs) to obtain a homogeneous set H of size k. It is easy to show that H is homogeneous relative to COL.

We now see how large n must be so that the construction be carried out. Note that $|V_i|$ will be divided in half $\leq i$ times in stage i. Hence $|V_{i+1}| \geq \frac{|V_i|}{2^i}$. Therefore

$$|V_i| \geq \frac{n}{2^{i^2/2}}.$$

We want $|V_{R(2,k)}| \geq 1$.

$$\frac{n}{2^{R(2,k)^2/2}} \geq 1$$

We do not use the 2 in the denominator of the exponent. We take $n = 2^{R(2,k)^2}$. By Theorem 2.7 we obtain

$$R(3,k) \le 2^{R(2,k)^2} \le 2^{2^{(2k-\Omega(\log k))^2}} \le 2^{4k-\Omega(\log k)}.$$

▮

Note 6.2 The proof of Theorem 6.1 generalizes to c-colors yielding

$$R(3,k,c) \le c^{c^{(ck-(c-1))-1}}.$$

This proof uses Note 2.9 for the bound on $R(2,k,c)$.

We now consider the a-ary Ramsey Theorem.

Theorem 6.3

1. *For all $a \ge 2$, for all k, $R(a,k) \le 2^{R(a-1,k)^{a-1}}$.*
2. *For all $a \ge 1$, for all k, $R(a,k) \le \mathrm{TOW}(a-1, 4k - \Omega(\log k))$.*

Proof:

1) Assume that $R(a-1,k)$ exists and $a \ge 2$.

CONSTRUCTION

$$\begin{aligned} x_1 &= 1 \\ \vdots &= \vdots \\ x_{a-2} &= a-2 \\ V_{a-2} &= [n] - \{x_1, \ldots, x_{a-2}\} \text{ We start indexing here for convenience.} \end{aligned}$$

Let $2 \leq i \leq R(a-1,k)$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^{**} : \binom{\{x_1,\ldots,x_{i-1}\}}{2} \rightarrow [2]$ are defined.

$$\begin{aligned} x_i = &\quad \text{the least element of } V_{i-1} \\ V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).} \end{aligned}$$

We define $COL^{**}(A \cup \{x_i\})$ for every $A \in \binom{\{x_1,\ldots,x_{i-1}\}}{a-2}$. We will also define smaller and smaller sets V_i.

For $A \in \binom{\{x_1,\ldots,x_{i-1}\}}{a-2}$

1. $COL^* : V_i \rightarrow [2]$ is defined by $COL^*(y) = COL(A \cup \{x_i, y\})$.

2. Let V_i be redefined as the largest homogeneous set for COL^*. Note that $|V_i|$ is at least $1/2$ of its previous size. (Recall that COL^* is a coloring of vertices, not edges, and that a homogenous set is one that is all the same color.)

3. $COL^{**}(A \cup \{x_i\})$ is the color of V_i.

We have

$$x_1, x_2, \ldots, x_{R(a-1,k)}$$

and every $(a-1)$-set is colored with one of 2 colors by COL^**. Apply the $(a-1)$-ary Ramsey Theorem to obtain a homogenous set. This is the homogenous set for the original coloring that we seek.

We now see how large n must be so that the construction be carried out. Note that $|V_i|$ will be divided by 2 at most $\binom{i}{a-2} \leq i^{a-2}$ times in stage i. Hence $|V_{i+1}| \geq \frac{|V_i|}{2^{i^{a-2}}}$. Therefore

$$|V_i| \geq \frac{n}{2^{i^{a-1}}}.$$

We want $|V_{R(a-1,k)}| \geq 1$.

$$\frac{n}{2^{R(a-1,k)^{a-1}}} \geq 1.$$

Hence $n = 2^{R(a-1,k)^{a-1}}$ suffices. Therefore

$$R(a,k) \leq 2^{R(a-1,k)^{a-1}}.$$

2) We prove this by induction on a. We do the base case for $a = 1, 2, 3$ so that the reader can see why the theorem gives the values that it does.

Base Case:

1. $a = 1$: $R(1,k) = 2k-1 = \mathrm{TOW}(0, 2k-1)$.

2. $a = 2$: $R(2,k) \leq 2^{2k-\Omega(\log k)} \leq \mathrm{TOW}(1, 2k - \Omega(\log k)) \leq \mathrm{TOW}(1, 4k - \Omega(\log k))$. by Theorem 2.7. We did not use the recurrence from Part 1; it would have give the bound $\mathrm{TOW}(1, 4k - \Omega(\log k))$.

3. $a = 3$: $R(3,k) \leq 2^{R(a-1,k)^2} \leq \mathrm{TOW}(2, 4k - \Omega(\log k))$ from Theorem 6.1. This proof does use the recurrence. The exponent of $a-1$ (2 in this case) goes directly into the top exponent which is why we end up with $4k - \Omega(\log k)$ as the top term instead of $2k$.

Induction Step: We assume the theorem is true for $a-1$. We may assume $a \geq 4$.

$$\begin{aligned}
R(a,k) &\leq 2^{R(a-1,k)^{a-1}} \text{ By Part 1.} \\
&\leq 2^{\mathrm{TOW}(a-2,4k-\Omega(\log k)^{a-1}} \text{ By the induction hypothesis.} \\
&\leq 2^{\mathrm{TOW}(a-2,4k-\Omega(\log k))} \text{ By Lemma 4.1.} \\
&\leq \mathrm{TOW}(a-1, 4k - \Omega(\log k)) \text{ By the definition of TOW.}
\end{aligned}$$

∎

Note 6.4 The proof of Theorem 6.3 generalizes to c-colors to yield

$$R(a, k, c) \leq \mathrm{TOW}_c(a-1, 2ck).$$

7 Conlon-Fox-Sudakov

Recall the following high level description of the Erdos-Rado proof:

- Color an edge by using PHP. This cuts the number of nodes in half. This is done $2^{R(2,k)}$ times.
- After *all* the edges of a complete graph are colored we use Ramsey's theorem. This will cut the number of nodes down by a log.

Every time we colored an edge we cut the number of vertices in half. Could we color fewer edges? Consider the following scenario:

$COL^{**}(x_1, x_2) = R$ and $COL^{**}(x_1, x_3) = B$. Intuitively the edge from x_2 to x_3 might not be that useful to us. *Therefore we will not color that edge!*

Two questions come to mind:

Question: How will we determine which edges are potentially useful?
Answer: We will associate to each x_i a string σ_i that keeps track of which edges (x_j, x_i) are colored, and if so what they are colored. (We denote the empty string by λ.) Say we already have

$$x_1, \ldots, x_i$$

$$\sigma_1, \ldots, \sigma_i$$

(where $\sigma_1 = \lambda$) defined. If $i < j$ then the edge (x_i, x_j) will be colored if σ_i and σ_j do not conflict.

Question: Since we only color some of the edges how will we use Ramsey's theorem?

Answer: We will not. Instead we go until one of the subscripts has either k R's or k B's in it.

We need two lemmas.

Lemma 7.1 *For all $k \geq 1$, for all $x \geq 0$, $\sum_{L=0}^{k-1} \binom{x+L}{L} = \binom{x+k}{k-1} = \binom{x+k}{x+1}$.*

Proof: We prove this by induction on k.

If $k = 1$ then we get $\binom{x}{0} = \binom{x+1}{0}$ which is true since they are both 1.

Assume true for $k-1$. So

$$\sum_{L=0}^{k-2} \binom{x+L}{L} = \binom{x+k-1}{k-2}.$$

Add $\binom{x+k-1}{k-1}$ to both sides

$$\sum_{L=0}^{k-1} \binom{x+L}{L} = \binom{x+k-1}{k-2} + \binom{x+k-1}{k-1} = \binom{x+k}{k-1}.$$

▮

Lemma 7.2 *Let $S \subseteq \{R, B\}^*$ be such that no string in S has $\geq k$ R's or $\geq k$ B's. Then*

$$\sum_{\sigma \in S} |\sigma| \leq A\sqrt{k}2^k$$

where $A = \frac{2}{\sqrt{\pi}} \sim 1.128$.

Proof:

Let $S_{i,j}$ be the set of all strings with i R's and j B's. Clearly

$$\sum_{\sigma \in S} |\sigma| = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \sum_{\sigma \in S_{i,j}} |\sigma|.$$

Note that every $\sigma \in S_{i,j}$ is of length $i + j$. Hence

$$\sum_{\sigma\in S_{i,j}} |\sigma| = \sum_{\sigma\in S_{i,j}} i+j = (i+j)\sum_{\sigma\in S_{i,j}} 1 = (i+j)|S_{i,j}| = (i+j)\binom{i+j}{j}.$$

Hence we need

$$\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}(i+j)\binom{i+j}{j} = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} i\binom{i+j}{j} + j\binom{i+j}{i} = 2\sum_{i=0}^{k-1} i\sum_{j=0}^{k-1}\binom{i+j}{j}$$

By Lemma 7.1 with $x = i$ and $L = j$ we obtain

$$\sum_{j=0}^{k-1}\binom{i+j}{j} = \binom{i+k}{i+1}.$$

Hence we seek

$$2\sum_{i=0}^{k-1} i\binom{i+k}{i+1}.$$

$$\sum_{i=0}^{k-1} i\binom{i+k}{i+1} = \sum_{i=0}^{k-1}(i+1)\binom{i+k}{i+1} - \sum_{i=0}^{k-1}\binom{i+k}{i+1}$$

We look at each piece separately.

$$(i+1)\binom{i+k}{i+1} = (i+1)\frac{(i+k)!}{(k-1)!(i+1)!} = \frac{(i+k)!}{(k-1)!i!} = k\frac{(i+k)!}{k!i!} = k\binom{i+k}{i}$$

Hence

$$\sum_{i=0}^{k-1}\binom{i+k}{i+1} = \sum_{i=0}^{k-1} k\binom{i+k}{i} = k\sum_{i=0}^{k-1}\binom{i+k}{i}$$

By Lemma 7.1 with $x = k$ and $L = i$ we obtain

$$\sum_{i=0}^{k-1} = \binom{i+k}{i} = \binom{2k}{k+1},$$

Putting all of this together and using Stirling's Formula we obtain

$$2\sum_{i=0}^{k-1} i\binom{i+k}{i+1} = 2\left(k\binom{2k}{k+1} - \sum_{i=0}^{k-1}\binom{i+k}{i+1}\right) \le 2k\binom{2k}{k} \le A\sqrt{k}2^{2k}$$

where $A = \frac{2}{\sqrt{\pi}} \sim 1.128$. ▮

Note 7.3 According to WolframAlpha [1]

$$\sum_{i=0}^{k-1}\binom{i+k}{i+1} = \left(1+\frac{1}{k}\right)\binom{2k}{k+1} + 1.$$

This would yield

$$\sum_{\sigma\in S}|\sigma| = 2\left(\left(k-1-\frac{1}{k}\right)\binom{2k}{k-1} - 1\right).$$

This more exact result does not help us obtain a smaller upper bound.

The following proof is by Conlon-Fox-Sudakov [3].

Theorem 7.4 *For all k, $R(3,k) \le 2^{A\sqrt{k}2^{2k}}$ where $A = \frac{2}{\sqrt{\pi}} \sim 1.128$.*

Proof: Let n be a number to be determined. Let COL be a 2-coloring of K_n^3. We define a sequence of vertices $x_1, x_2, \ldots$ and a sequence of strings $\sigma_1, \sigma_2, \ldots \in \{R, B, Q\}^*$. We will stop at the first i such that $|\sigma_i|$ has either k R's or k B's. (We will later show that this must happen.) We say that two strings *are compatible* if there is no position where one says R and one says B.

FILL IN- WE MIGHT USE SUBSET OF INSTEAD OF COMPATIBLE.

Recall the definition of a homogeneous set relative to a coloring of a 1-hypergraph from the note following Definition 2.3. We will use it here.

Here is the intuition: Let $x_1 = 1$ and $x_2 = 2$. Let $\sigma_1 = \lambda$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be a homogeneous set of size at least $\frac{n-2}{2}$. We will only work with V_1 from now on. Let $COL^{**}(x_1, x_2)$ be the color of V_1. Let $\sigma_1 = COL^{**}(x_1, x_2)$. So far this looks like the proof given in Theorem 6.1

Let x_3 be the least vertex in V_1. The number x_3 induces *two* colorings of $V_1 - \{x\}$:

$$COL^*_{1,3}(y) = COL(x_1, x_3, y)$$

$$COL^*_{2,3}(y) = COL(x_2, x_3, y)$$

Let V_2 be a homogeneous set relative to $COL^*_{1,3}$ of size $\frac{|V_1|-1}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. We also set $\sigma_3 = COL^{**}(x_1, x_3)$, though we will append to σ_3 later. Restrict $COL^*_{2,3}$ to elements of V_2, though still call it $COL^*_{2,3}$. We will only work with V_2 from now on.

Will we color (x_2, x_3)? If σ_2 and σ_3 are compatible then YES. If not then we won't. This is the key- every time we color an edge we divide V in half. We will not always color and edge- only the promising ones. Hence V will not decrease as quickly as was done in the proof of Theorem 6.1.

If σ_2 and σ_3 are compatible then we let V_2 (reuse var name) be a homogeneous set relative to $COL^*_{2,3}$ of size at least $\frac{|V_2|}{2}$. Let $COL^{**}(x_2, x_3)$ be the color of V_2. Append that color to σ_3 to form a new σ_2.

If σ_2 and σ_3 are not compatible then we do color (x_2, x_3). We do however append a Q to σ_2. If $\sigma_3 = RQ$ this means that the edge between x_1 and x_3 is colored R and there is no edge between x_2 and x_3.

We describe the construction formally.

CONSTRUCTION

$$
\begin{aligned}
V_1 = &\ \ [n] \\
x_1 = &\ \ 1 \\
x_2 = &\ \ 2 \\
\sigma_1 = &\ \ \lambda \\
\sigma_2 = &\ \ \lambda \text{ (This will change.)} \\
(\forall y \in V_1 - X_1)[COL^*(y) = &\ \ COL(x_1, x_2, y)] \\
V_2 = &\ \ \text{the largest homogeneous set for } COL^* \\
COL^{**}(x_1, x_2) = &\ \ \text{the color of } V_2 \\
\sigma_2 = &\ \ \sigma_2 COL^{**}(x_1, x_2)
\end{aligned}
$$

KEY: for all $y \in V_1$, $COL(x_1, x_2, y) = COL^{**}(x_1, x_2)$.

Let $i \geq 2$, and assume that V_{i-1}, $x_1, \ldots, x_{i-1}$, $\sigma_1, \ldots, \sigma_{i-1}$ are defined. Also assume that $COL^{**} : \binom{\{x_1, \ldots, x_{i-1}\}X}{2} \rightarrow [2]$ is partially defined (that is, there may be some pairs that are not assigned a color). If σ_{i-1} has either k R's or k $B's$ then stop. Otherwise proceed.

$$
\begin{aligned}
\sigma_i = &\ \ \lambda \text{ (This will change.)} \\
x_i = &\ \ \text{the least element of } V_{i-1} \\
V_i = &\ \ V_i - \{x_i\} \text{ (We will change this set without changing its name.)}
\end{aligned}
$$

We define *some of* $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, …, $COL^{**}(x_i, x_i)$. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

For $j = 0$ to $i - 1$

1. If σ_j and σ_i are compatible then proceed, otherwise append Q to σ_i to form new σ_i and go to next j.

2. $COL^* : V_i \rightarrow [2]$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.

3. V_i is the largest homogeneous set for COL^*. Note that $|V_i|$ will be decreased by at most half.

4. $COL^{**}(x_j, x_i)$ is the color of V_i.

5. Append $COL^{**}(x_j, x_i)$ to the end of σ_i to form a new σ_i.

KEY: for all $y \in V_i$, for all $1 \le a < b \le i$ such that $\sigma_a \preceq \sigma_b$, $COL(x_a, x_b, y) = COL^{**}(x_a, x_b)$.

END OF CONSTRUCTION

It is not clear that the construction ends; however, that will be proven by Claim 1 below which is also used to obtain the upper bound on $R(3, k)$. For now we assume that the construction ends.

When the construction ends we have a σ that has either k R's or k B's. We assume that its R's. Let

$$\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$$

be all of the prefixes of σ that end in an R. By the construction, for all $1 \le a < b < c \le k$

$$COL(\sigma_{i_a}, \sigma_{i_b}, \sigma_{i_c}) = R.$$

Hence the set

$$x_{i_1}, \ldots, x_{i_k}$$

forms a homogeneous set of size k.

We now need to determine a bound on n.

Def 7.5 If $\sigma \in \{R, B, Q\}^*$ then $\mathrm{squash}(\sigma)$ is formed by removing all of the Q's from σ.

Claim 1: For all $i_1 < i_2$, $\mathrm{squash}(\sigma_{i_1}) \neq \mathrm{squash}(\sigma_{i_2})$.

Proof of Claim 1: Assume, by way of contradiction, that $i_1 < i_2$ and $\mathrm{squash}(\sigma_{i_1}) = \mathrm{squash}(\sigma_{i_2})$.

Mini Claim: For $1 \le j \le i_1 - 1$ $\sigma_{i_1}[j] = \sigma_{i_2}[j]$. We prove this by induction on j.

Proof of Mini Claim:

Base Case: $j = 1$. $\sigma_{i_1}[1]$ and $\sigma_{i_2}[1]$ are non Q elements. They must be the same since $\mathrm{squash}(\sigma_{i_1}) = \mathrm{squash}(\sigma_{i_2})$.

Induction Step: Assume that $\sigma_{i_1}[1, 2, \ldots, j-1] = \sigma_{i_2}[1, 2, \ldots, j-1]$. We also assume $j \le i_1 - 1$. Since $\sigma_{i_1}[1, 2, \ldots, j-1] = \sigma_{i_2}[1, 2, \ldots, j-1]$ they will either both get a Q for the jth symbol (in which case the Mini Claim holds for j) or both get a non-Q. If they both get a non-Q it must be the same symbol since $\sigma_{i_1}[1, 2, \ldots, j-1] = \sigma_{i_2}[1, 2, \ldots, j-1]$ and $\mathrm{squash}(\sigma_{i_1}) = \mathrm{squash}(\sigma_{i_2})$.

End of Proof of Mini Claim

By the Mini Claim σ_{i_1} is a prefix of σ_{i_2}. Hence when σ_{i_2} is being defined and $j = i_1$, either an R or a B will be appended to σ_{i_2} (say R). At that point $\sigma_{i_1}R$ will be a prefix of σ_{i_2}, so $\mathrm{squash}(\sigma_{i_1}) \ne \mathrm{squash}(\sigma_{i_2})$. This is a contradiction.

End of Proof of Claim

Let S, T be defined as follows.

$$T = \{\sigma_1, \sigma_2, \ldots\}.$$

$$S = \{\mathrm{squash}(\sigma) : \sigma \in T\}.$$

Since the construction ends when σ_i has either k R's or k B's, we have that $S \subseteq \{R, B\}^{\le 2k-1}$ and hence is finite. Therefore the construction will terminate.

How big is T? Actually, this is not what we care about. What we care about is how many times the number of vertices will be cut in half. For every non-Q element in σ_i, the number of vertices gets cut in half. So we really care about

$$\sum_{\sigma \in T} \text{Number of non-}Q \text{ in } \sigma \ .$$

By Claim 1 the summation can be restated (without fear of double counting) as

$$\sum_{\tau \in S} |\tau|.$$

By Lemma 7.2 this is bounded by $A\sqrt{k}2^k$. Therefore we need $\frac{n}{2^{A\sqrt{k}2^{2k}}} \geq 1$. Thus we can take $n \geq 2^{A\sqrt{k}2^{2k}}$. Hence $R(3,k) \leq 2^{A\sqrt{k}2^{2k}}$. ∎

Note 7.6 The proof of Theorem 7.4 generalizes to c-colors yielding

$$R(3,k,c) \leq c^{c^2 k^2 c^{ck}}.$$

The proof of this result does not use a modified Lemma 7.2. A modified Lemma 7.2 might lead to a better result. Instead we use a modified Lemma 7.8. We leave the proof of this bound as an exercise for the reader.

We consider the a-ary Ramsey Theorem. We need a lemma that is analogous to Lemma 7.2; however, it won't be as exact.

Def 7.7 We denote a colored graph by (V, E, COL) where V is the vertices, E is the edges and COL is the coloring of the edges. We assume that if $|V| = n$ then $V = [n]$.

Lemma 7.8 *Let S be the subset of 2-colored complete $(a-1)$-hypergraphs that have no homogenous set of size k.*

$$\sum_{(V,E,COL) \in S} |E| \leq R(a,k)^a 2^{R(a,k)^{a-1}}.$$

Proof:

The largest size of V such that a colored $(a-1)$-hypergraph (V,E) has no homogenous set of size k is bounded above by $R(a-1,k)$. Hence what we want to bound.

$$\sum_{i=1}^{R(a-1,k)} \sum_{(V,E,COL)\in S,|V|=i} |E| \le \sum_{i=1}^{R(a-1,k)} \sum_{(V,E,COL)\in S,|V|=i} i^{a-1}$$

The number of 2-colored a-hypergraphs on i vertices is bounded above by 2^{i^a}. Hence we can bound the above sum by

$$\sum_{i=1}^{R(a-1,k)} 2^{i^{a-1}} i^{a-1} \le R(a-1,k) 2^{R(a-1,k)^{a-1}} R(a-1,k)^{a-1} \le R(a-1,k)^a 2^{R(a-1,k)^{a-1}}$$

▮

Theorem 7.9 *For all $a \ge 3$ the following holds.*

1. *For all k, $R(a,k) \le 2^{R(a-2,k)^{a-1} 2^{R(a-2,k)^{a-2}}}$.*

2. $$R(a,k) \le \begin{cases} \mathrm{TOW}(a-1, 2k+O(\log k) & \textit{if } a \textit{ is odd} \\ \mathrm{TOW}(a-1, 4k-\Omega(\log k) & \textit{if } a \textit{ is even} \end{cases}$$

Proof:

1)

CONSTRUCTION

$$
\begin{aligned}
V_{a-2} = & \ [n] \text{ We start indexing here for convenience.} \\
x_1 = & \ 1 \\
x_2 = & \ 2 \\
\vdots = & \ \vdots \\
x_{a-1} = & \ a-1 \\
X_{a-1} = & \ \{x_1, \ldots, x_{a-1}\} \\
G_1 = & \ ([1], \emptyset) \\
G_2 = & \ ([2], \emptyset) \\
\vdots & \ \vdots \\
G_{a-2} = & \ ([a-2], \emptyset) \\
G_{a-1} = & \ ([a-1], \emptyset) \text{ (The edge set will change.)} \\
(\forall y \in V_{a-1} - X_{a-1})[COL^*(y) = & \ COL(x_1, x_2, \ldots, x_{a-1}, y)] \\
V_{a-1} = & \ \text{ the largest homogeneous set for } COL^* \\
COL^{**}(x_1, \ldots, x_{a-1}) = & \ \text{ the color of } V_{a-1} \\
G_{a-1} = & \ ([a-1], \{[a-1]\}) \text{ with } COL_{a-1}([a-1]) = COL^* * (x_1, \ldots, x_{a-1})
\end{aligned}
$$

The G_i's will be 2-colored $(a-2)$-hypergraphs. We say that two such objects are *compatible* if there is no edge where the disagree on the color. There may be vertices and edges in one and not the other.

KEY: for all $y \in V_1$, $COL(x_1, \ldots, x_{a-1}, y) = COL^{**}(x_1, \ldots, x_{a-1})$.

Let $i \geq 2$, and assume that V_i, $X_i = \{x_1, \ldots, x_i\}$, and $\sigma_1, \ldots, \sigma_i$ are defined. Also assume that $COL^{**} : \binom{X}{a-1} \rightarrow [2]$ is partially defined (that is, there may be some $(a-1)$-sets that are not assigned a color). If G_i has a homogenous set of size k then stop. Otherwise proceed.

$$
\begin{aligned}
G_i = &\ \emptyset \text{ (This will change.)} \\
x_i = &\ \text{the least element of } V_i \\
V_i = &\ V_i - \{x_i\} \text{ (We will change this set without changing its name.)}
\end{aligned}
$$

We define *some of* $COL^{**}(A \cup \{x_i\})$ for $A \in \binom{X_i}{a-2}$. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

FILL IN- COMPATIBLE- MIGHT USE SUBGRAPH

For $A \in \binom{X}{a-2}$

1. If for every $j \in A$, G_j and G_i are compatible then proceed, otherwise go to next A. (Note that in Theorem 7.4 we explicitly colored the point Q to mean that we are not coloring it. Here we simply do not color it.)

2. $COL^* : V_i \rightarrow [2]$ is defined by $COL^*(y) = COL(A \cup \{x_i, y\})$.

3. V_i is the largest homogeneous set for COL^*. Note that $|V_i|$ will be at most divided by 2.

4. $COL^{**}(A \cup \{x_i\})$ is the color of V_i.

5. In G_i color A with the color of V_i.

END OF CONSTRUCTION

KEY: for all $y \in V_i$, for all $B \in \binom{X_i}{a-1}$, $COL(B \cup \{y\}) = COL^{**}(B)$.

It is not clear that the construction ends; however, that will be proven by Claim 3 below which is also used to obtain the recurrence for $R(a, k)$ in terms of $R(a-2, k)$. For now we assume that the construction ends.

When the construction ends we have a G that has a homogenous set of size k. We assume that its color is R. The homogenous set of G is clearly a homogenous set for the original coloring COL.

We now need to determine a bound on n.

How long can this sequence

$$S = \{G_1, G_2, \ldots\}.$$

go on for? Actually, this is not what we care about. What we care about is how many times the number of times the number of vertices will be cut by $1/2$. Determining this well take a sequence of claims.

Claim 1: Each G_i in the construction is a complete graph.

Proof of Claim 1:

FILL I PROOF

End of Proof of Claim 1

Def 7.10 If G is a complete 2-colored graph on vertex set $1 \le i_1 < i_2 < \cdots i_L$ then $\mathrm{squash}(G)$ is formed by, for $1 \le j \le L$, renaming vertices i_j by j.

Claim 2: For all $i_1 < i_2$, $\mathrm{squash}(G_{i_1}) \neq \mathrm{squash}(G_{i_2})$.

Proof of Claim 2:

FILL I PROOF

End of Proof of Claim 2

Claim 3: The construction works with $n =$ FILL IN LATER- RECURRENCE

Proof of Claim 3:

FILL IN PROOF

End of Proof of Claim 3

FILL IN- MIGHT NEED TO CHANGE THE REST OF THE PROOF.

By Claim 3 $R(a-2,k)^{a-1}2^{R(a-2,k)^{a-2}}$. FILL IN - MIGHT NEET TO CHANGE THIS Hence it will suffice to take $n = 2^{R(a-2,k)^{a-1}2^{R(a-2,k)^{a-2}}}$. Therefore

$$R(a,k) \le 2^{R(a-2,k)^{a-1} 2^{R(a-2,k)^{a-2}}}.$$

It will be sometimes be convenient to use the alternative form:

$$R(a,k) \le 2^{2^{R(a-2,k)^{a-2}} + (a-1)\lg(R(a-2,k))}.$$

2) We prove this by induction on a. We do the base case for $a = 1, 2, 3, 4, 5$ so that the reader can see why the theorem has different cases for a even and a odd.

Base Case:

1. $R(1,k) = 2k - 1 \le 2k = \mathrm{TOW}(0, 2k) \le \mathrm{TOW}(0, (4+\epsilon)k)$.

2. $R(2,k) \le 2^{2k} = \mathrm{TOW}(1, 2k)$. This follows from Theorem 2.7.

3. $R(3,k) \le 2^{A\sqrt{k}2^{2k}} \le 2^{2^{(2k+O(\log k)}} = \mathrm{TOW}(2, 2k + O(\log k))$. This follows from Theorem 7.4.

4. $R(4,k) \le 2^{2^{R(2,k)^2} + 3\lg(R(2,k))} \le 2^{2^{2^{4k-\Omega(\log k)}} + 6k} \le 2^{2^{2^{(4k-\Omega(\log k)}}} = \mathrm{TOW}(3, 4k - \Omega(\log k)$. Note that we get the $(4k - \Omega(\log k)$ term since (simplifying) $(2^{2k})^2 = 2^{4k}$. Contrast this with what happens when $a = 5$.

5. $R(5,k) \le 2^{2^{R(3,k)^3} + 4\lg(R(3,k))}$. Note that

   $$R(3,k)^3 \le (2^{2^{2k+O(\log k)}})^3 = (2^{3 \times 2^{(2k+O(\log k))}}) \le 2^{2^{2k+O(\log k)}} = \mathrm{TOW}(2, 2k + O(\log k)).$$

   Hence

   $$2^{R(3,k)^3} \le \mathrm{TOW}(3, 2k + O(\log k))$$

   and

   $$\lg(R(3,k)) \le \mathrm{TOW}(1, 2k + O(\log k)).$$

Hence FILL IN LATER TO GET $\mathrm{TOW}(4, 2k + O(\log k))$. Note that we get a a $(2k + O(\log(k))k$ term because (simplifying) $(2^{2^{(Bk)}})^3 = 2^{3\times 2^{Bk}}$. Hence the cubing does not affect the top term.

Because $R(3,k)$ ended up with a top term of $(2+\epsilon)k$ every even $k \geq 2$ will end up with a top term of $(2+\epsilon)k$. Because $R(4,k)$ ended up with a top term of $(4+\epsilon)k$ every even $k \geq 2$ will end up with a top term of $(4+\epsilon)k$.

Induction Step: Assume the theorem is true for all $a' < a$. We can also assume $a \geq 6$. We do the case where a is odd. The case where a is even is similar.

By the recurrence, the above, and Lemma 4.1, for almost all k,

$$\begin{aligned}
R(a,k) &\leq 2^{R(a-2,k)^{a-1}2^{R(a-2,k)^{a-2}}} \text{ By the recurrence from Part 1.} \\
&\leq 2^{\mathrm{TOW}(a-3,2k+O(\log k))^{a-1}2^{\mathrm{TOW}(a-3,2k+O(\log k))^{a-3}}} \text{ By the Induction Hypothesis.} \\
&\leq 2^{\mathrm{TOW}(a-3,2k+O(\log k))2^{\mathrm{TOW}(a-3,2k+O(\log k))}} \text{ By Lemma 4.1.2} \\
&\leq 2^{\mathrm{TOW}(a-3,2k+O(\log k))\mathrm{TOW}(a-2,2k+O(\log k))} \text{ By the definition of TOW} \\
&\leq 2^{\mathrm{TOW}(a-2,2k+O(\log k)} \text{ By Lemma 4.1.} \\
&\leq \mathrm{TOW}(a-2, 2k+O(\log k)) \text{ By the definition of TOW.}
\end{aligned}$$

Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon/3$ to obtain the result we seek. ∎

FILL IN- FIGUREOUT

Note 7.11 The proof of Theorem 7.9 generalizes to c-colors yielding the following.

1. For all k, $R(a,k,c) \leq c^{R(a-2,k,c)^{a-1}c^{R(a-2,k,c)^{a-2}}}$.

2. For all $\epsilon > 0$, for almost all k,

$$R(a,k,c) \le \begin{cases} \mathrm{TOW}_c(a-1,(c+\epsilon)k) & \text{if } a \text{ is odd} \\ \mathrm{TOW}_c(a-1,(2c+\epsilon)k) & \text{if } a \text{ is even} \end{cases}$$

We leave this as an exercise for the reader.

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