1 Definitions from Topology

Def 1.1

1. $X$ is a metric space if there exists a function $d$ (called a metric) with the following properties. (1) $d(x, y) = 0$ iff $x = y$, (2) $d(x, y) = d(y, x)$, (3) $d(x, y) \leq d(x, z) + d(z, y)$ (this is called the triangle inequality).

2. $X, Y$ metric space. If $x \in X$ and $\epsilon \in \mathbb{R}^+$ then $B(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$. Sets of this form are called balls.

3. Any union of balls is an open set.

4. If $A$ is the complement of an open set then $A$ is closed.

5. Let $A \subseteq X$ and $x \in X$. $x$ is a limit point of $X$ if $(\forall \epsilon > 0)(\exists y) [y \in B(x, \epsilon) \cap A]$.

6. If $x_1, x_2, \ldots \in X$ then $\lim_i x_i = x$ means $(\forall \epsilon > 0)(\exists i)(\forall j)[j \geq i \Rightarrow x_j \in B(x, \epsilon)]$.

7. $T : X \rightarrow X$. $T$ is continuous if for all $x, x_1, x_2, \ldots \in X \lim_i x_i = x \Rightarrow \lim_i f(x_i) = f(x)$.

8. $T : X \rightarrow X$ is unif-continous if $(\forall \epsilon)(\exists \delta)(\forall a, b \in X)[d(a, b) < \delta \Rightarrow d(T(a), T(b)) < \epsilon]$.

9. $T : X \rightarrow X$ is bi-unif-continous if $T$ is a bijection, $T$ is uniformly continous, and $T^{-1}$ is uniformly continous.

10. If $A \subseteq X$ then the closure of $X$, denoted $cl(A)$, is the intersection of all closed sets containing $X$.

11. $X$ is Barg if every infinite subset of $X$ has a limit point.

Fact 1.2

1. $cl(A)$ is the smallest closed set containing $A$.

2. If a set is closed then it contains all of its limit points.

3. In a metric space $cl(A)$ is the union of $A$ and the limit points of $A$.

4. If $X$ is Barg and $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ are nonempty closed sets then $\cap_i X_i \neq \emptyset$. 

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**Def 1.3** Let $X$ be a metric space and $T : X \to X$ be continuous. Let $x \in X$. The point $x$ is **Recurrent for $T$** if

$$(\forall \epsilon)(\exists n)[d(T^{(n)}(x), x) < \epsilon].$$

We prove a theorem about Recurrent points and then apply it to get VDW theorem.

**Def 1.4** A metric space $S$ is **minimal** if, for every $x \in S$,

$$S = cl(\{\ldots, T^{-2}(x), T^{-1}(x), T^0(x), T^1(x), T^2(x), T^3(x), \ldots\})$$

We show the following by a multiple induction.

1. $A_r$: $(\forall \epsilon > 0)(\exists x, y \in S, n \in \mathbb{N})$
   $$d(T^{(n)}(x), y) < \epsilon, d(T^{(2n)}(x), y) < \epsilon, \ldots, d(T^{(rn)}(x), y) < \epsilon.$$

2. $B_r$: $(\forall \epsilon > 0)(\forall z \in S)(\exists x \in S, n \in \mathbb{N})$
   $$d(T^{(n)}(x), z) < \epsilon, d(T^{(2n)}(x), z) < \epsilon, \ldots, d(T^{(rn)}(x), z) < \epsilon.$$

3. $C_r$: $(\forall \epsilon > 0)(\forall z \in S)(\exists x \in S, n \in \mathbb{N}, e' > 0)$
   $$T^{(n)}(B(x, e')) \subseteq B(z, \epsilon), T^{(2n)}(B(x, e')) \subseteq B(z, \epsilon), \ldots, T^{(rn)}(B(x, e')) \subseteq B(z, \epsilon).$$

4. $(\forall \epsilon > 0)(\exists w \in S, n \in \mathbb{N})$
   $$d(T^{(n)}(w), w) < \epsilon, d(T^{(2n)}(w), w) < \epsilon, \ldots, d(T^{(rn)}(w), w) < \epsilon.$$

**Def 1.5** Let $X$ be a metric space, $T : X \to X$ be a bijection, and $x \in X$.

1. $CLT(x) = cl(\{\ldots, T^{-3}(x), T^{-2}(x), T^{-1}(x), T^0(x), T^1(x), T^2(x), T^3(x), \ldots\})$

2. $x$ is **homogenous** if
   $$(\forall y \in CLT(x))[CLT(x) = CLT(y)].$$

3. $X$ is **barg** if every infinite subset of $X$ has a limit point in $X$.

**Lemma 1.6** Let $(X, \preceq)$ be a partial order. If every chain has an upper bound then there exists a maximal element

**Lemma 1.7** Let $X$ be a metric space, $T : X \to X$ be bi-continuous, and $x \in X$. If $y \in CLT(x)$ then $CLT(y) \subseteq CLT(x)$.

**Theorem 1.8** Let $X$ be a barg metric space. Let $T : X \to X$ be a bijection then there exists a homogenous point $x \in X$.

**Theorem 1.9** For all $c$, for all $k$, for every $c$-coloring of $\mathbb{Z}$ there exists a monochromatic arithmetic sequence of length $k$. 

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