The Monotone Sequence Game
Exposition by Gasarch

1 Introduction

This is a writeup of some of the material in [**?**].

Recall the following theorem. For six proofs of this theorem see [**?**].

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Def 1.1 Let $n \geq 1$. Let L be any linear order. Let $\vec{a} \in L^*$. A *monotonic sub sequence of* $\vec{a}$ *of length* n (henceforth n-mono-subseq) is a sub sequence that is either increasing or decreasing.

Theorem 1.2 *Let* $n \geq 1$*. Let* L *be any linear order with at least* $(n-1)^2+1$ *elements. Let* $\vec{a}$ *be a sequence of at least* $(n-1)^2+1$ *distinct elements from* L*. Then either there exists an* n*-mono-subseq.*

This theorem inspires the following game.

Def 1.3 Let $n \geq 1$. Let L be a linear order.

1. Let $G(L,n)$ be the following game. Players I and II alternate play with I going first. In each turn a Player picks an element of L that has not been picked before. The picks forms a sequence. The first Player to complete an n-mono-subseq wins. If L is finite and all of the numbers are chosen without a winner, then the game is a tie.
2. Let $\vec{a} \in L^*$. Let $GAL(L,n,\vec{a})$ be the game that is just like $GAL(L,n)$ but it starts with position $\vec{a}$. Player I has the first move iff $|\vec{a}|$ is even. Note that if $\vec{a}$ is the empty vector then we recover $GAL(L,n)$.

Def 1.4 Let $n \geq 1$. Let L be a linear order. Let $\vec{a} \in L^*$.

$$WIN(L,n,\vec{a}) = \begin{cases} I & \text{if Player I has a winning strategy for the game } G(L,n,\vec{a}) \ ; \\ II & \text{if Player II has a winning strategy for the game } G(L,n,\vec{a}) \ ; \\ T & \text{if neither Player has a winning strategy for the game } G(L,n,\vec{a}) \ . \end{cases} \quad (1)$$

Note that if $WIN(L,n,\vec{a}) = T$ and both Players play perfectly then the game is a TIE.

Notation 1.5 $WIN(L,n)$ is $WIN(L,n,\lambda)$ where λ is the empty vector.

Theorem 1.6 *Let L be a linear order such that $|L| \geq (n-1)^2 + 1$. Then $WIN(L,n) \neq T$.*

Proof: This follows from Theorem 1.2. ∎

Def 1.7 If $N \in \mathsf{N}$ then L_N is the ordering $1 < 2 < \cdots < N$. As usual Z is the integers N is the naturals, Q is the rationals. These are all ordered sets.

Note 1.8 By Theorem 1.6 $W(L_{(n-1)^2+1}, n) \neq T$. The following question is open and interesting: Given n, what is the least m such that $W(L_m, n) \neq T$?

We show the following.

Def 1.9

1. For all $N \in \mathsf{N}$ there exists $n_0 \in \mathsf{N}$ and $J \in \{I, II\}$ such that

$$(\forall n \geq n_0)[WIN(L_N, n) = J].$$

2. For all $n \geq 4$, $WIN(Q,n) = I$.

2 Useful Definitions and Lemmas

Def 2.1 Let L be a linear order.

1. A function $f : L \to L$ is *an order preserving bijection* if f is a bijection and, for all $x < y \in L$, $f(x) < f(y)$.
2. A function $f : L \to L$ is *an order investing bijection* if f is a bijection and, for all $x < y \in L$, $f(x) > f(y)$.

3 $WIN(L_N, n)$

4 $WIN(\mathsf{Q}, n)$

We leave the following easy theorem as an exercise.

Theorem 4.1 $W(\mathsf{Q}, 1) = I$, $W(\mathsf{Q}, 2) = II$, $W(\mathsf{Q}, 3) = I$.

Lemma 4.2 *Assume the following are true. Let* $\vec{a} \in \mathsf{Q}^*$ *and* $n \in \mathsf{N}$. *Let* a_i *be the* i*th element of* $\vec{a}$. *Let* $\vec{a}'$ *be* $\vec{a}$ *with* a_i *removed.*

1. $W(L, n, \vec{a}) = II$.
2. $W(L, n, \vec{a}') = II$.
3. *At the end of the game* $W(L, n, \vec{a})$ *there is an* n*-mono-subseq that does not contain* a_i.

Then $W(L, n, \vec{a}) = I$. *(This yields a contradiction.)*

Theorem 4.3 *For all* $n \geq 4$, $W(\mathsf{Q}, n) = I$.

Proof: By Theorem 1.6 one of the two Players has a winning strategy. Assume, by way of contradiction, that II has a winning strategy. We give a strategy for Player I such that, if Player II plays his winning strategy, Player I wins.
Winning strategy for Player I

1. On the first move Player I plays a_1 (the value of a_1 does not matter).
2. Player II's plays a_2. We assume that $a_1 < a_2$. (If $a_2 < a_1$ then a similar strategy works.)
3. Player I plays $a_3 < a_1 < a_2$.
4. There are four cases depending on what Player II does.

(a) Player II plays $a_4 < a_3 < a_1 < a_2$. If $n = 4$ then Player I plays $a_5 < a_4$ to form $a_1 > a_2 > a_4 > a_5$ and win.

If $n \geq 5$ then Player I plays a_5 such that

$$a_4 < a_3 < a_5 < a_1 < a_2.$$

We show that the premises of Lemma 4.2 hold. Let $\vec{a} = (a_1, a_2, a_3, a_4, a_5)$ and $i = 1$. Since Player II was playing a winning strategy $WIN(L, n, \vec{a}) = II$. Look at $\vec{\hat{a}} = (a_2, a_3, a_4, a_5)$. Note that

$$a_4 < a_3 < a_5 < a_2.$$

Claim 1: At the end of the game there will be an n-mono-subseq that does not contain a_1.

Proof of Claim 1:

If a_1 is in an increasing subsequence then that subsequence looks like

$$a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n$$

where $i_2 \geq 2$ and $i_3 \geq 6$. Hence the following is an increasing subsequence of length n that does not contain a_1.

$$a_3 < a_5 < a_{i_3} < \cdots < a_{i_n}.$$

If a_1 is in a decreasing subsequence then that subsequence looks like

$$a_1 > a_{i_2} > a_{i_3} > a_{i_4} > \cdots > a_{i_n}$$

where $i_2 \geq 3$. Hence the following is a decreasing subsequence of length n that does not contain a_1.

$$a_2 > a_{i_2} > a_{i_3} > a_{i_4} > \cdots > a_{i_n}$$

End of Proof of Claim 1

(b) Player II plays a_4 such that $a_3 < a_4 < a_1 < a_2$. **Claim 2:** At the end of the game there will be an n-mono-subseq that does not contain a_1.

Proof of Claim 2:

If a_1 is in an increasing subsequence then that subsequence looks like:

$$a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n$$

where $i_3 \geq 5$. Hence the following is an increasing subsequence of length n that does not have a_1.

$$a_3 < a_4 < a_{i_3} < \cdots < a_{i_n}.$$

If a_1 is in a decreasing subsequence then that subsequence looks like:

$$a_1 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n.$$

Since $a_2 > a_1$ we know $i_2 \geq 3$. Hence the following is a decreasing subsequence of length n that does not have a_1.

$$a_2 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

End of Proof of Claim 2

(c) Player II plays a_4 such that $a_3 < a_1 < a_4 < a_2$.

Claim 3: At the end of the game there will be an n-mono-subseq that does not contain a_1.

Proof of Claim 3:

If a_1 is in an increasing subsequence then that subsequence looks either like

$$a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n$$

where $i_3 \geq 5$

Hence the following is an increasing subsequence of length n that does not have a_1.

$$a_3 < a_4 < a_{i_3} < \cdots a_{i_4}.$$

If a_1 is in a decreasing subsequence then that subsequence looks like

$$a_1 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

where $i_2 \geq 3$.

Hence we have the following decreasing subsequence of length n that does not have a_1.

$$a_2 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

$$1 > i_2 > i_3 > \cdots > i_n$$

End of Proof of Claim 3

(d) Player II plays a_4 such that $a_3 < a_1 < a_2 < a_4$. If $n = 4$ then Player I plays $a_5 > a_4$ and wins via $a_1 < a_2 < a_4 < a_5$. If $n \geq 5$ then Player I plays a_5 such that

$$a_3 < a_5 < a_1 < a_2 < a_4.$$

Claim 4: At the end of the game there will be an n-mono-subseq that does not contain a_3.

Proof of Claim 4:

If a_3 is in an increasing subsequence then that subsequence looks either like

$$a_3 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n$$

where $i_2 \geq 4$

Hence the following is an increasing subsequence of length n that does not have a_3.

$$a_1 < a_2 < a_{i_3} < \cdots a_{i_4}.$$

If a_3 is in a decreasing subsequence then that subsequence looks like

$$a_{i_1} > a_3 > a_{i_3} > \cdots > a_{i_n}$$

$$i_1 < 3 < i_3 < \cdots < i_n$$

where $i_3 \geq 6$.

Hence we have the following decreasing subsequence of length n that does not have a_3.

$$a_{1_1} > a_5 > a_{i_3} > \cdots > a_{i_n}$$

End of Proof of Claim 4

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