The Monotone Sequence Game
Exposition by Gasarch

1 Introduction

This is a writeup of some of the material in [?].

Recall the following theorem. For six proofs of this theorem see [?].

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Def 1.1 Let \( n \geq 1 \). Let \( L \) be any linear order. Let \( \vec{a} \in L^* \). A monotonic subsequence of \( \vec{a} \) of length \( n \) (henceforth \( n \)-mono-subseq) is a subsequence that is either increasing or decreasing.

Theorem 1.2 Let \( n \geq 1 \). Let \( L \) be any linear order with at least \((n - 1)^2 + 1\) elements. Let \( \vec{a} \) be a sequence of at least \((n - 1)^2 + 1\) distinct elements from \( L \). Then either there exists an \( n \)-mono-subseq.

This theorem inspires the following game.

Def 1.3 Let \( n \geq 1 \). Let \( L \) be a linear order.

1. Let \( G(L, n) \) be the following game. Players I and II alternate play with I going first. In each turn a Player picks an element of \( L \) that has not been picked before. The picks forms a sequence. The first Player to complete an \( n \)-mono-subseq wins. If \( L \) is finite and all of the numbers are chosen without a winner, then the game is a tie.

2. Let \( \vec{a} \in L^* \). Let \( GAL(L, n, \vec{a}) \) be the game that is just like \( GAL(L, n) \) but it starts with position \( \vec{a} \). Player I has the first move iff \( |\vec{a}| \) is even. Note that if \( \vec{a} \) is the empty vector then we recover \( GAL(L, n) \).

Def 1.4 Let \( n \geq 1 \). Let \( L \) be a linear order. Let \( \vec{a} \in L^* \).

\[
WIN(L, n, \vec{a}) = \begin{cases} 
I & \text{if Player I has a winning strategy for the game } G(L, n, \vec{a}) \\
II & \text{if Player II has a winning strategy for the game } G(L, n, \vec{a}) \\
T & \text{if neither Player has a winning strategy for the game } G(L, n, \vec{a}) 
\end{cases}
\]  

(1)

Note that if \( WIN(L, n, \vec{a}) = T \) and both Players play perfectly then the game is a TIE.
**Notation 1.5** $WIN(L, n)$ is $WIN(L, n, \lambda)$ where $\lambda$ is the empty vector.

**Theorem 1.6** Let $L$ be a linear order such that $|L| \geq (n-1)^2 + 1$. Then $WIN(L, n) \neq T$.

**Proof:** This follows from Theorem 1.2. \[\square\]

**Def 1.7** If $N \in \mathbb{N}$ then $L_N$ is the ordering $1 < 2 < \cdots < N$. As usual $\mathbb{Z}$ is the integers, $\mathbb{N}$ is the naturals, $\mathbb{Q}$ is the rationals. These are all ordered sets.

**Note 1.8** By Theorem 1.6 $W(L_{(n-1)^2+1}, n) \neq T$. The following question is open and interesting: Given $n$, what is the least $m$ such that $W(L_m, n) \neq T$?

We show the following.

**Def 1.9**

1. For all $N \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ and $J \in \{I, II\}$ such that

\[\forall n \geq n_0 [WIN(L_N, n) = J].\]

2. For all $n \geq 4$, $WIN(\mathbb{Q}, n) = I$.

## 2 Useful Definitions and Lemmas

**Def 2.1** Let $L$ be a linear order.

1. A function $f : L \to L$ is an order preserving bijection if $f$ is a bijection and, for all $x < y \in L$, $f(x) < f(y)$.

2. A function $f : L \to L$ is an order investing bijection if $f$ is a bijection and, for all $x < y \in L$, $f(x) > f(y)$. 
We leave the following easy theorem as an exercise.

**Theorem 4.1** $W(Q, 1) = I, W(Q, 2) = II, W(Q, 3) = I$.

**Lemma 4.2** Assume the following are true. Let $\vec{a} \in Q^*$ and $n \in N$. Let $a_i$ be the $i$th element of $\vec{a}$. Let $\vec{a}'$ be $\vec{a}$ with $a_i$ removed.

1. $W(L, n, \vec{a}) = II$.
2. $W(L, n, \vec{a}') = II$.
3. At the end of the game $W(L, n, \vec{a})$ there is an $n$-mono-subseq that does not contain $a_i$.

Then $W(L, n, \vec{a}) = I$. (This yields a contradiction.)

**Theorem 4.3** For all $n \geq 4$, $W(Q, n) = I$.

**Proof:** By Theorem 1.6 one of the two Players has a winning strategy. Assume, by way of contradiction, that $II$ has a winning strategy. We give a strategy for Player I such that, if Player II plays his winning strategy, Player I wins.

**Winning strategy for Player I**

1. On the first move Player I plays $a_1$ (the value of $a_1$ does not matter).
2. Player II’s plays $a_2$. We assume that $a_1 < a_2$. (If $a_2 < a_1$ then a similar strategy works.)
3. Player I plays $a_3 < a_1 < a_2$.
4. There are four cases depending on what Player II does.
(a) Player II plays \( a_4 < a_3 < a_1 < a_2 \). If \( n = 4 \) then Player I plays \( a_5 < a_4 \) to form \( a_1 > a_2 > a_4 > a_5 \) and win.

If \( n \geq 5 \) then Player I plays \( a_5 \) such that

\[
a_4 < a_3 < a_5 < a_1 < a_2.
\]

We show that the premises of Lemma 4.2 hold. Let \( \vec{a} = (a_1, a_2, a_3, a_4, a_5) \) and \( i = 1 \). Since Player II was playing a winning strategy \( WIN(L, n, \vec{a}) = II \). Look at \( \vec{a}' = (a_2, a_3, a_4, a_5) \). Note that

\[
a_4 < a_3 < a_5 < a_2.
\]

**Claim 1:** At the end of the game there will be an \( n \)-mono-subseq that does not contain \( a_1 \).

**Proof of Claim 1:**

If \( a_1 \) is in an increasing subsequence then that subsequence looks like

\[
a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}
\]

\[
1 < i_2 < i_3 < \cdots < i_n
\]

where \( i_2 \geq 2 \) and \( i_3 \geq 6 \). Hence the following is an increasing subsequence of length \( n \) that does not contain \( a_1 \).

\[
a_3 < a_5 < a_{i_3} < \cdots < a_{i_n}.
\]

If \( a_1 \) is in a decreasing subsequence then that subsequence looks like

\[
a_1 > a_{i_2} > a_{i_3} > a_{i_4} > \cdots > a_{i_n}
\]

where \( i_2 \geq 3 \). Hence the following is a decreasing subsequence of length \( n \) that does not contain \( a_1 \).

\[
a_2 > a_{i_2} > a_{i_3} > a_{i_4} > \cdots > a_{i_n}
\]

**End of Proof of Claim 1**
(b) Player II plays $a_4$ such that $a_3 < a_4 < a_1 < a_2$. **Claim 2:** At the end of the game there will be an $n$-mono-subseq that does not contain $a_1$.

**Proof of Claim 2:**

If $a_1$ is in an increasing subsequence then that subsequence looks like:

$$a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$

where $i_3 \geq 5$. Hence the following is an increasing subsequence of length $n$ that does not have $a_1$.

$$a_3 < a_4 < a_{i_3} < \cdots < a_{i_n}.$$ 

If $a_1$ is in a decreasing subsequence then that subsequence looks like:

$$a_1 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n.$$ 

Since $a_2 > a_1$ we know $i_2 \geq 3$. Hence the following is a decreasing subsequence of length $n$ that does not have $a_1$.

$$a_2 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}.$$ 

**End of Proof of Claim 2**

(c) Player II plays $a_4$ such that $a_3 < a_1 < a_4 < a_2$.

**Claim 3:** At the end of the game there will be an $n$-mono-subseq that does not contain $a_1$.

**Proof of Claim 3:**

If $a_1$ is in an increasing subsequence then that subsequence looks either like

$$a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$
$1 < i_2 < i_3 < \cdots < i_n$

where $i_3 \geq 5$

Hence the following is an increasing subsequence of length $n$ that does not have $a_1$.

$$a_3 < a_4 < a_{i_3} < \cdots a_{i_n}.$$  

If $a_1$ is in a decreasing subsequence then that subsequence looks like

$$a_1 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

where $i_2 \geq 3$.

Hence we have the following decreasing subsequence of length $n$ that does not have $a_1$.

$$a_2 > a_{i_2} > a_{i_3} > \cdots > a_{i_n}$$

$$1 > i_2 > i_3 > \cdots > i_n$$

**End of Proof of Claim 3**

(d) Player II plays $a_4$ such that $a_3 < a_1 < a_2 < a_4$. If $n = 4$ then Player I plays $a_5 > a_4$ and wins via $a_1 < a_2 < a_4 < a_5$. If $n \geq 5$ then Player I plays $a_5$ such that

$$a_3 < a_5 < a_1 < a_2 < a_4.$$  

**Claim 4:** At the end of the game there will be an $n$-mono-subseq that does not contain $a_3$.

**Proof of Claim 4:**

If $a_3$ is in an increasing subsequence then that subsequence looks either like

$$a_3 < a_{i_2} < a_{i_3} < \cdots < a_{i_n}$$

$$1 < i_2 < i_3 < \cdots < i_n$$
where $i_2 \geq 4$
Hence the following is an increasing subsequence of length $n$ that does not have $a_3$.

$$a_1 < a_2 < a_{i_3} < \cdots < a_{i_4}.$$ 

If $a_3$ is in a decreasing subsequence then that subsequence looks like

$$a_{i_1} > a_3 > a_{i_3} > \cdots > a_{i_n}$$ 

$$i_1 < 3 < i_3 < \cdots < i_n$$ 

where $i_3 \geq 6$.
Hence we have the following decreasing subsequence of length $n$ that does not have $a_3$.

$$a_{i_1} > a_5 > a_{i_3} > \cdots > a_{i_n}$$

**End of Proof of Claim 4**