Showing that a Propositional Logic is Incomplete

1 Kleene’s System

Kleene proposed the following set of axioms and rule of inference for Propositional Logic.

AXIOMS

For any formulas $p, q, r$ the following are axioms.

1. $p \Rightarrow (q \Rightarrow p)$.
2. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.
3. $(p \land q) \Rightarrow p$.
4. $(p \land q) \Rightarrow q$.
5. $p \Rightarrow (q \Rightarrow (p \land q))$.
6. $p \Rightarrow (p \lor q)$.
7. $q \Rightarrow (p \lor q)$.
8. $(p \Rightarrow q) \Rightarrow ((r \Rightarrow q) \Rightarrow ((p \lor r) \Rightarrow q))$.
9. $(p \Rightarrow q) \Rightarrow ((p \Rightarrow \neg q) \Rightarrow (\neg p))$.
10. $\neg \neg p \Rightarrow p$

RULES OF INFERENCE

Modus Ponens. That is, if you have $p \Rightarrow q$ and $p$ then you get $q$.

COMPLETENESS It is known that Kleene’s system is complete— any tautology is provable.

2 Heyting’s System

Heyting was an intuitionist. Roughly speaking this means that he didn’t believe that $(p \lor \neg p)$ is true.

Hence he wanted a system that was NOT complete. He wanted a system where you COULD NOT prove $(p \lor \neg p)$.

His system is just like Kleene’s except that he replaced the last axiom with $\neg p \Rightarrow (p \Rightarrow q)$.

3 How to Prove Incompleteness?

We will show that Heyting’s system cannot prove $(p \lor \neg p)$ by using INVARIANT— we will show that the AXIOMS have a certain property, the Rules of inference preserve that property but the statement $(p \lor \neg p)$ does not have that property.

Def 3.1 We define truth tables for $\land, \lor, \neg$ that allow the truth values $0, \frac{1}{2}, 1$. 

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1. $p \land q$ evaluates to $\min\{p, q\}$.

2. $p \lor q$ evaluates to $\max\{p, q\}$.

3. $\neg p$ evaluates to $1 - p$.

4. $p \Rightarrow q$ is evaluated by the following table which is a natural extrapolation of the usual rules.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>$p \Rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Def 3.2** A formula is a taut $\forall$ if, for any truth setting of 0’s, $\frac{1}{2}$’s, and 1’s, we get 1.

**Theorem 3.3**

1. All of the axioms of Heyting’s system are taut $\forall$.

2. If $p$ is a taut $\forall$ and $p \Rightarrow q$ is a taut $\forall$ then $q$ is a taut $\forall$ (so Modus Ponens preserves taut $\forall$).

3. $(p \lor \neg p)$ is not a taut $\forall$.

4. $(p \lor \neg p)$ cannot be derived in Heyting’s system.

**Proof:**

1) This is a case analysis which we defer to the next section.

2) Assume $p$ is a taut $\forall$ and $p \Rightarrow q$ is a taut $\forall$. We assume that $p, q$ use the same set of vars. We show that $q$ is a taut $\forall$. Let $s$ be any setting of the vars in $q$ to $\{0, \frac{1}{2}, 1\}$. Under this setting $p$ evaluates to 1 and $p \Rightarrow q$ evaluates to 1. Since $p \Rightarrow q$ evaluates to 1 we must have $p \leq q$. Since $p$ evaluates to 1, $q$ evaluates to 1.

3) In $(p \lor \neg p)$ look at the setting $p = \frac{1}{2}$. Then $\neg p$ evaluates to $\frac{1}{2}$, and the $\lor$ of two $\frac{1}{2}$ is $\frac{1}{2}$. Hence there is a setting where $(p \lor \neg p)$ evaluates to $\frac{1}{2} \neq 1$.

4) Since all of the axioms are taut $\forall$ and Modus Ponens preserves this, any formula that can be derived is a tautology $\forall$. Since $(p \lor \neg p)$ is not a taut $\forall$, it cannot be derived.

\[ \square \]
4 The Axioms are taut+

We show each axiom is a taut+ by trying to find a setting where it evaluates to 0 or $\frac{1}{2}$ and failing. The case where it evaluates to $\frac{1}{2}$ often splits into two cases since there are 2 ways that $(p \Rightarrow q)$ can evaluate to $\frac{1}{2}$.

Often we will find we are forced to have the variables be in $\{0, 1\}$. In this case we will stop since it is already known that the axioms are tautologies in the usual sense.

1. $(q \Rightarrow (p \Rightarrow q))$
   
   (a) Evaluates to 0. Then $q = 1$ and $(p \Rightarrow q) = 0$. Hence $p = 1$ and $q = 0$, so $p, q \in \{0, 1\}$.
   
   (b) Evaluates to $\frac{1}{2}$. There are two cases.
      
      i. $q = \frac{1}{2}$ and $(p \Rightarrow q) = 0$. Since $(p \Rightarrow q) = 0$ we have $q = 0$ which contradicts $q = \frac{1}{2}$.
      
      ii. $q = 1$ and $(p \Rightarrow q) = \frac{1}{2}$ In order for $(p \Rightarrow q) = \frac{1}{2}$ you must have $q \in \{0, \frac{1}{2}\}$, which contradicts $q = 1$.

2. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.
   
   (a) Evaluates to 0. Then $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = 0$. Hence $(p \Rightarrow r) = 0$. Hence $p = 1$ and $r = 0$. Since the expression evaluates to 1 we must have $(p \Rightarrow (q \Rightarrow r)) = 1$.
   
   Since $p = 1$ we must have $(q \Rightarrow r) = 1$. Since $r = 0$ we must have $q = 0$. We have $p, q, r \in \{0, 1\}$.
   
   (b) Evaluates to $\frac{1}{2}$. There are two cases.
      
      i. $(p \Rightarrow (q \Rightarrow r)) = \frac{1}{2}$ and $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = 0$. The later forces $p = 1$, $r = 0$, and from $p = 1$ we get $q = 1$. We have $p, q, r \in \{0, 1\}$.
      
      ii. $(p \Rightarrow (q \Rightarrow r)) = 1$ and $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = \frac{1}{2}$. We have two cases based on why $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = \frac{1}{2}$.
         
         A. $(p \Rightarrow q) = \frac{1}{2}$ and $(p \Rightarrow r) = 0$. The latter implies that $r = 0$ and $p = 1$. With this, the former implies $q = \frac{1}{2}$. With these values $(p \Rightarrow (q \Rightarrow r))$ is $(1 \Rightarrow (\frac{1}{2} \Rightarrow 0))$ which is $(1 \Rightarrow \frac{1}{2}) = \frac{1}{2} \neq 1$ So we’re done.
         
         B. $(p \Rightarrow q) = 1$ and $(p \Rightarrow r) = \frac{1}{2}$. Since $(p \Rightarrow r) = \frac{1}{2}$ we have $p > r$.

(c) $(p \land q) \Rightarrow p$.

(d) $(p \land q) \Rightarrow q$.

(e) $p \Rightarrow (q \Rightarrow (p \land q))$.

(f) $p \Rightarrow (p \lor q)$.

(g) $q \Rightarrow (p \lor q)$.

(h) $(p \Rightarrow q) \Rightarrow ((r \Rightarrow q) \Rightarrow ((p \lor r) \Rightarrow q))$.

(i) $(p \Rightarrow q) \Rightarrow ((p \Rightarrow \neg q) \Rightarrow (\neg p))$. 

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