Showing that a Propositional Logic is Incomplete

1 Kleene's System

Kleene proposed the following set of axioms and rule of inference for Propositional Logic.
AXIOMS

For any formulas p, q, r the following are axioms.

1. $p \Rightarrow (q \Rightarrow p)$.
2. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.
3. $(p \wedge q) \Rightarrow p$.
4. $(p \wedge q) \Rightarrow q$.
5. $p \Rightarrow (q \Rightarrow (p \wedge q))$.
6. $p \Rightarrow (p \vee q)$.
7. $q \Rightarrow (p \vee q)$.
8. $(p \Rightarrow q) \Rightarrow ((r \Rightarrow q) \Rightarrow ((p \vee r) \Rightarrow q))$.
9. $(p \Rightarrow q) \Rightarrow ((p \Rightarrow \neg q) \Rightarrow (\neg p))$.
10. $\neg\neg p \Rightarrow p$

RULES OF INFERENCE

Modus Ponens. That is, if you have $p \Rightarrow q$ and p then you get q.

COMPLETENESS It is known that Kleene's system is complete— any tautology is provable.

2 Heyting's System

Heyting was an intuitionist. Roughly speaking this means that he didn't believe that $(p \vee \neg p)$ is true.

Hence he *wanted* a system that was NOT complete. He wanted a system where you COULD NOT prove $(p \vee \neg p)$.

His system is just like Kleene's except that he replaced the last axiom with
$\neg p \Rightarrow (p \Rightarrow q)$.

3 How to Prove Incompleteness?

We will show that Heyting's system cannot prove $(p \vee \neg p)$ by using INVARIANTS— we will show that the AXIOMS have a certain property, the Rules of inference preserve that property but the statement $(p \vee \neg p)$ does not have that property.

Def 3.1 We define truth tables for $\wedge, \vee, \neg$ that allow the truth values $0, \frac{1}{2}, 1$.

1. $p \wedge q$ evaluates to $\min\{p,q\}$.
2. $p \vee q$ evaluates to $\max\{p,q\}$.
3. $\neg p$ evalutes to $1-p$.
4. $p \Rightarrow q$ is evaluated by the following table which is a natural extrapolation of the usual rules.

p	q	$p \Rightarrow q$
$-$	$-$	$-$
0	0	1
0	$\frac{1}{2}$	1
0	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	1	1
1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1

Def 3.2 A formula is a taut$^{+}+$ if, for any truth setting of 0's, $\frac{1}{2}$'s, and 1's, we get 1.

Theorem 3.3

1. *All of the axioms of Heyting's system are* taut^{+}.
2. *If p is a* taut^{+}*and $p \Rightarrow q$ is a* taut^{+}*then q is a* taut^{+}*(so Modus Ponens preserves* taut^{+}*).*
3. $(p \vee \neg p)$ *is not a* taut^{+}.
4. $(p \vee \neg p)$ *cannot be derived in Heyting's system.*

Proof:
1) This is a case analysis which we defer to the next section.

2) Assume p is a taut^{+}and $p \Rightarrow q$ is a taut^{+}. We assume that p, q use the same set of vars. We show that q is a taut^{+}. Let s be any setting of the vars in q to $\{0, \frac{1}{2}, 1\}$. Under this setting p evaluates to 1 and $p \Rightarrow q$ evaluates to 1. Since $p \Rightarrow q$ evaluates to 1 we must have $p \leq q$. Since p evaluates to 1, q evaluates to 1.

3) In $(p \vee \neg p)$ look at the setting $p = \frac{1}{2}$. Then $\neg p$ evaluates to $\frac{1}{2}$, and the $\vee$ of two $\frac{1}{2}$ is $\frac{1}{2}$. Hence there is a setting where $(p \vee \neg p)$ evaluates to $\frac{1}{2} \neq 1$.

4) Since all of the axioms are taut^{+}and Modus Ponens preserves this, any formula that can be derived is a tautalogy+. Since $(p \vee \neg p)$ is not a taut^{+}, it cannot be derived.

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4 The Axioms are taut$^+$

We show each axioms is a taut+ by trying to find a setting where it evaluates to 0 or $\frac{1}{2}$ and failing. The case where if evaluates to $\frac{1}{2}$ often splits into two cases since there are 2 ways that $(p \Rightarrow q)$ can evaluate to $\frac{1}{2}$.

Often we will find we are forced to have the variables be in $\{0, 1\}$. In this case we will stop since it is already known that the axioms are tautologies in the usual sense.

1. $(q \Rightarrow (p \Rightarrow q))$
   (a) Evaluates to 0. Then $q = 1$ and $(p \Rightarrow q) = 0$. Hence $p = 1$ and $q = 0$, so $p, q \in \{0, 1\}$.
   (b) Evaluates to $\frac{1}{2}$. There are two cases.
      i. $q = \frac{1}{2}$ and $(p \Rightarrow q) = 0$. Since $(p \Rightarrow q) = 0$ we have $q = 0$ which contradicts $q = \frac{1}{2}$)
      ii. $q = 1$ and $(p \Rightarrow q) = \frac{1}{2}$ In order for $(p \Rightarrow q) = \frac{1}{2}$ you must have $q \in \{0, \frac{1}{2}\}$, which contradicts $q = 1$.

2. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.
   (a) Evaluates to 0. Then $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = 0$. Hence $(p \Rightarrow r) = 0$. Hence $p = 1$ and $r = 0$. Since the expression evaluates to 1 we must have $(p \Rightarrow (q \Rightarrow r)) = 1$. Since $p = 1$ we must have $(q \Rightarrow r) = 1$. Since $r = 0$ we must have $q = 0$. We have $p, q, r \in \{0, 1\}$.
   (b) Evaulates to $\frac{1}{2}$. There are two cases.
      i. $(p \Rightarrow (q \Rightarrow r)) = \frac{1}{2}$ and $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = 0$. The later forces $p = 1$, $r = 0$, and from $p = 1$ we get $q = 1$. We have $p, q, r \in \{0, 1\}$.
      ii. $(p \Rightarrow (q \Rightarrow r)) = 1$ and $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = \frac{1}{2}$. We have two cases based on why $((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) = \frac{1}{2}$.
         A. $(p \Rightarrow q) = \frac{1}{2}$ and $(p \Rightarrow r) = 0$. The latter implies that $r = 0$ and $p = 1$. With this, the former implies $q = \frac{1}{2}$. With these values $(p \Rightarrow (q \Rightarrow r))$ is $(1 \Rightarrow (\frac{1}{2} \Rightarrow 0))$ which is $(1 \Rightarrow \frac{1}{2}) = \frac{1}{2} \neq 1$ So we're done.
         B. $(p \Rightarrow q) = 1$ and $(p \Rightarrow r) = \frac{1}{2}$. Since $(p \Rightarrow r) = \frac{1}{2}$ we have $p > r$.
   (c) $(p \wedge q) \Rightarrow p$.
   (d) $(p \wedge q) \Rightarrow q$.
   (e) $p \Rightarrow (q \Rightarrow (p \wedge q))$.
   (f) $p \Rightarrow (p \vee q)$.
   (g) $q \Rightarrow (p \vee q)$.
   (h) $(p \Rightarrow q) \Rightarrow ((r \Rightarrow q) \Rightarrow ((p \vee r) \Rightarrow q))$.
   (i) $(p \Rightarrow q) \Rightarrow ((p \Rightarrow \neg q) \Rightarrow (\neg p))$.