Roth's Theorem: If $A \subseteq [n]$ is large then it has a 3-AP
Roth's Proof
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1 Roth's Theorem

Notation 1.1 Let $[n] = \{1, \ldots, n\}$. If $k \in \mathsf{N}$ then k-AP means an arithmetic progression of size k.

Consider the following statement:
If $A \subseteq [n]$ and $\#(A)$ is 'big' then A must have a 3-AP.

This statement, made rigorous, is true. In particular, the following is true and easy:
Let $n \geq 3$. If $A \subseteq [n]$ and $\#(A) \geq 0.7n$ then A must have a 3-AP.

Can we lower the constant 0.7? We can lower it as far as we like if we allow n to start later: Roth [3, 4, 5] proved the following using analytic means.
$(\forall\lambda > 0)(\exists n_0 \in \mathsf{N})(\forall n \geq n_0)(\forall A \subseteq [n])[\#(A) \geq \lambda n \Rightarrow A$ has a 3-AP$]$.

The analogous theorem for 4-APs was later proven by Szemeredi [3, 6] by a combinatorial proof. Szemeredi [7] later (with a much harder proof) generalized from 4 to any k.

We prove the $k = 3$ case using the analytic techniques of Roth; however, we rely heavily on Gowers [2, 1]

Definition 1.2 Let $sz(n)$ be the least number such that, for all $A \subseteq [n]$, if $\#(A) \geq sz(n)$ then A has a 3-AP. Note that if $A \subseteq [a, a+n-1]$ and $\#(A) \geq sz(n)$ then A has a 3-AP. Note also that if $A \subseteq \{a, 2a, 3a, \ldots, na\}$ and $\#(A) \geq sz(n)$ then A has a 3-AP. More generally, if A is a subset of any equally spaced set of size n, and $\#(A) \geq sz(n)$, then A has a 3-AP.

2 Sparse Intervals

The next lemma states that if A is 'big' and 3-free then it is somewhat uniform. There cannot be sparse intervals of A. The intuition is that if A has a sparse interval then the rest of A has to be dense to make up for it, and it might have to be so dense that it has a 3-AP.

Lemma 2.1 *Let $n, n_0 \in \mathsf{N}; \lambda, \lambda_0 \in (0, 1)$. Assume $\lambda < \lambda_0$ and $(\forall m \geq n_0)[sz(m) \leq \lambda_0 m]$. Let $A \subseteq [n]$ be a 3-free set such that $\#(A) \geq \lambda n$. Let a, b be such that $a < b$, $a > n_0$, and $n - b > n_0$. Then $\lambda_0(b - a) - n(\lambda_0 - \lambda) \leq \#(A \cap [a, b])$.*

Proof:

Since A is 3-free and $a \geq n_0$ and $n - b \geq n_0$ we have $\#(A \cap [1, a-1]) < \lambda_0(a-1) < \lambda_0 a$ and $\#(A \cap [b+1, n]) < \lambda_0(n - b)$. Hence

$$\begin{aligned} \lambda n \leq \#(A) = &\ \#(A \cap [1, a-1]) + \#(A \cap [a, b]) + \#(A \cap [b+1, n]) \\ \lambda n \leq &\ \lambda_0 a + \#(A \cap [a, b]) + \lambda_0(n - b) \\ \lambda n - \lambda_0 n + \lambda_0 b - \lambda_0 a \leq &\ \#(A \cap [a, b]) \\ \lambda_0(b - a) - n(\lambda_0 - \lambda) \leq &\ \#(A \cap [a, b]). \end{aligned}$$

∎

3 Notation

Throughout this paper the following hold.

1. $n \in \mathsf{N}$ is a fixed large prime.
2. $\mathsf{Z}_n = \{1, \ldots, n\}$ with modular arithmetic.
3. $\omega = e^{2\pi i/n}$.
4. If a is a complex number then $|a|$ is its length.
5. If A is a set then $|A|$ is its cardinality.

4 Counting 3-AP's

Lemma 4.1 *Let $A, B, C \subseteq [n]$. The number of $(x, y, z) \in A \times B \times C$ such that $x + z \equiv 2y \pmod{n}$ is*

$$\frac{1}{n} \sum_{x,y,z \in [n]} A(x)B(y)C(z) \sum_{r=1}^{n} \omega^{-r(x-2y+z)}.$$

Proof:

We break the sum into two parts:
Part 1:

$$\frac{1}{n} \sum_{x,y,z \in [n], x+z \equiv 2y \pmod{n}} A(x)B(y)C(z) \sum_{r=1}^{n} \omega^{-r(x-2y+z)}.$$

Note that we can replace $\omega^{-r(x-2y+z)}$ with $\omega^0 = 1$. We can then replace $\sum_{r=1}^{n} 1$ with n. Hence we have

$$\frac{1}{n} \sum_{x,y,z \in [n], x+z \equiv 2y \pmod{n}} A(x)B(y)C(z)n = \sum_{x,y,z \in [n], x+z \equiv 2y \pmod{n}} A(x)B(y)C(z)$$

This is the number of $(x, y, z) \in A \times B \times C$ such that $x + z \equiv 2y \pmod{n}$.
Part 2:

$$\frac{1}{n} \sum_{x,y,z \in [n], x+z \not\equiv 2y \pmod{n}} A(x)B(y)C(z) \sum_{r=1}^{n} \omega^{-r(x-2y+z)}.$$

We break this sum up depending on what the (nonzero) value of $w = x + z - 2y \pmod{n}$. Let

$$S_u = \sum_{x,y,z \in [n], x-2y+z=2} A(x)B(y)C(z) \sum_{r=1}^{n} \omega^{-ru}.$$

Since $u \neq 0$, $\sum_{r=1}^{n} \omega^{-ru} = \sum_{r=1}^{n} \omega^{-r} = 0$. Hence $S_u = 0$.
Note that

$$\frac{1}{n}\sum_{x,y,z\in[n],x+z\not\equiv 2y \pmod n} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)} = \frac{1}{n}\sum_{u=1}^{n-1} S_u = 0$$

The lemma follows from Part 1 and Part 2. ▮

Lemma 4.2 *Let $A \subseteq [n]$. Let $B = C = A \cap [n/3, 2n/3]$. The number of $(x,y,z) \in A \times B \times C$ such that x, y, z forms a 3-AP is at least*

$$\frac{1}{2n}\sum_{x,y,z\in[n]} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)} - O(n).$$

Proof: By Lemma 4.1

$$\frac{1}{n}\sum_{x,y,z\in[n]} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)}$$

is the number of $(x,y,z) \in A \times B \times C$ such that $x+z \equiv 2y \pmod n$. This counts three types of triples:

- Those that have $x = y = z$. There are $n/3$ of them.
- Those that have $x + z = 2y + n$. There are $O(1)$ of them.
- Those that have $x \neq y$, $y \neq z$, $x \neq z$, and $x + z = 2y$.

Hence

$$\#(\{(x,y,z) : (x+z = 2y) \wedge x \neq y \wedge y \neq z \wedge x \neq z\}) = \frac{1}{n}\sum_{x,y,z\in[n]} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)} - O(n).$$

We are not done yet. Note that $(5, 10, 15)$ may show up as $(15, 10, 5)$. Every triple appears at most twice. Hence
$\#(\{(x,y,z) : (x+z = 2y) \wedge x \neq y \wedge y \neq z \wedge x \neq z\})$
$\leq \quad 2\#(\{(x,y,z) : (x < y < z) \wedge (x+z = 2y) \wedge x \neq y \wedge y \neq z \wedge x \neq z\})$.
Therefore

$$\frac{1}{2n}\sum_{x,y,z\in[n]} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)} - O(n) \leq \text{ the number of 3-AP's with } x \in A,\ y \in B,\ z \in C\ .$$

▮

We will need to re-express this sum. For that we will use Fourier Analysis.

5 Fourier Analysis

Definition 5.1 If $f : \mathsf{Z}_n \to \mathsf{N}$ then $\hat{f} : \mathsf{Z}_n \to \mathsf{C}$ is

$$\hat{f}(r) = \sum_{s\in[n]} f(s)\omega^{-rs}.$$

$\hat{f}$ is called the *Fourier Transform* of f.

What does $\hat{f}$ tell us? We look at the case where f is the characteristic function of a set $A \subseteq [n]$. Henceforth we will use $A(x)$ instead of $f(x)$.

We will need the followng facts.

Lemma 5.2 *Let* $A \subseteq \{1, \ldots, n\}$.

1. $\hat{A}(n) = \#(A)$.
2. $\max_{r\in[n]} |\hat{A}(r)| = \#(A)$.
3. $A(s) = \frac{1}{n}\sum_{r=1}^{n} \hat{A}(r)\omega^{-rs}$. *DO WE NEED THIS?*
4. $\sum_{r=1}^{n} |\hat{A}(r)|^2 = n\#(A)$.
5. $\sum_{s=1}^{n} A(s) = \frac{1}{n}\sum_{r=1}^{n} \hat{A}(r)$.

Proof:

Note that $\omega^n = 1$. Hence

$$\hat{A}(n) = \sum_{s\in[n]} A(s)\omega^{-ns} = \sum_{s\in[n]} A(s) = \#(A).$$

Also note that

$$|\hat{A}(r)| = |\sum_{s\in[n]} A(s)\omega^{-rs}| \le \sum_{s\in[n]} |A(s)\omega^{-rs}| \le \sum_{s\in[n]} |A(s)||\omega^{-rs}| \le \sum_{s\in[n]} |A(s)| = \#(A).$$

▮

Informal Claim: If $\hat{A}(r)$ is large then there is an arithmetic sequence P with difference r^{-1} (mod n) such that $\#(A \cap P)$ is large.

We need a lemma before we can proof the claim.

Lemma 5.3 *Let* $n, m \in \mathsf{N}$, $s_1, \ldots, s_m$, *and* $0 < \lambda, \alpha, \epsilon < 1$ *be given (no order on* $\lambda, \alpha, \epsilon$ *is implied). Assume that* $(\lambda - \frac{m-1}{m}(\lambda + \epsilon)) \ge 0$. *Let* $f(x_1, \ldots, x_m) = |\sum_{j=1}^{m} x_j\omega^{s_j}|$. *The maximum value that* $f(x_1, \ldots, x_m)$ *can achieve subject to the following two constraints (1)* $\sum_{j=1}^{m} x_j \ge \lambda n$, *and (2)* $(\forall j)[0 \le x_i \le (\lambda + \epsilon)\frac{n}{m}]$ *is bounded above by* $\epsilon mn + (\lambda + \epsilon)\frac{n}{m}|\sum_{j=1}^{m} \omega^{s_j}|$

Proof:

Assume that the maximum value of f, subject to the constraints, is achieved at $(x_1, \ldots, x_m)$. Let MIN be the minimum value that any variable x_i takes on (there may be several variables that take this value). What is the smallest that MIN could be? By the contraints this would occur when all but one of the variables is $(\lambda+\epsilon)\frac{n}{m}$ and the remaining variable has value MIN. Since $\sum_{x_i} \geq \lambda n$ we have

$MIN + (m-1)(\lambda+\epsilon)\frac{n}{m} \geq \lambda n$

$MIN + \frac{m-1}{m}(\lambda+\epsilon)n \geq \lambda n$

$MIN \geq \lambda n - \frac{m-1}{m}(\lambda+\epsilon)n$

$MIN \geq (\lambda - \frac{m-1}{m}(\lambda+\epsilon))n$

Hence note that, for all j,

$x_j - MIN \leq x_j - (\lambda - \frac{m-1}{m}(\lambda+\epsilon))n$

Using the bound on x_j from constraint (2) we obtain

$$\begin{aligned} x_j - MIN &\leq (\lambda+\epsilon)\frac{n}{m} - (\lambda - \frac{m-1}{m}(\lambda+\epsilon))n \\ &\leq ((\lambda+\epsilon)\frac{1}{m} - (\lambda - \frac{m-1}{m}(\lambda+\epsilon)))n \\ &\leq ((\lambda+\epsilon)\frac{1}{m} - \lambda + \frac{m-1}{m}(\lambda+\epsilon))n \\ &\leq \epsilon n \end{aligned}$$

Note that

$$\begin{aligned} |\sum_{j=1}^m x_j\omega^{s_j}| = &\ |\sum_{j=1}^m (x_j - MIN)\omega^{s_j} + \sum_{j=1}^m MIN\omega^{s_j}| \\ \leq &\ |\sum_{j=1}^m (x_j - MIN)\omega^{s_j}| + |\sum_{j=1}^m MIN\omega^{s_j}| \\ \leq &\ \sum_{j=1}^m |(x_j - MIN)||\omega^{s_j}| + MIN|\sum_{j=1}^m \omega^{s_j}| \\ \leq &\ \sum_{j=1}^m \epsilon n + MIN|\sum_{j=1}^m \omega^{s_j}| \\ \leq &\ \epsilon mn + MIN|\sum_{j=1}^m \omega^{s_j}| \\ \leq &\ \epsilon mn + (\lambda+\epsilon)\frac{n}{m}|\sum_{j=1}^m \omega^{s_j}| \end{aligned}$$

▮

Lemma 5.4 *Let $A \subseteq [n]$, $r \in [n]$, and $0 < \alpha < 1$. If $|\hat{A}(r)| \geq \alpha n$ and $|A| \geq \lambda n$ then there exists $m \in \mathsf{N}$, $0 < \epsilon < 1$, and an arithmetic sequence P within Z_n, of length $\frac{n}{m} \pm O(1)$ such that $\#(A \cap P) \geq (\lambda+\epsilon)\frac{n}{m}$. The parameters ϵ and m will depend on λ and α but not n.*

Proof: Let m and ϵ be parameters to be picked later. We will note constraints on them as we go along. (Note that ϵ will not be used for a while.)

Let $1 = a_1 < a_2 < \cdots < a_{m+1} = n$ be picked so that

$a_2 - a_1 = a_3 - a_2 = \cdots = a_m - a_{m-1}$ and $a_{m+1} - a_m$ is as close to $a_2 - a_1$ as possible.

For $1 \leq j \leq m$ let

$$P_j = \{s \in [n] : a_j \leq rs \pmod{n} < a_{j+1}\}.$$

Let us look at the elements of P_j. Let r^{-1} be the inverse of r mod n.

1. s such that $a_j \equiv rs \pmod{n}$, that is, $s \equiv a_j r^{-1} \pmod{n}$.
2. s such that $a_j + 1 \equiv rs \pmod{n}$, that is $s \equiv (a_j+1)r^{-1} \equiv a_j r^{-1} + r^{-1} \pmod{n}$.

3. s such that $a_j + 2 \equiv rs \pmod n$, that is $s \equiv (a_j + 2)r^{-1} \equiv a_j r^{-1} + 2r^{-1} \pmod n$.

4. $\vdots$

Hence P_j is an arithmetic sequence within Z_n which has difference r^{-1}. Also note that $P_1, \ldots, P_m$ form a partition of Z_n into m parts of size $\frac{n}{m} + O(1)$ each.

Recall that

$$\hat{A}(r) = \sum_{s \in [n]} A(s)\omega^{-rs}.$$

Lets look at $s \in P_j$. We have that $a_j \le rs \pmod n < a_{j+1}$. Therefore the values of $\{\omega^{rs} : s \in P_j\}$ are all very close together. We will pick $s_j \in P_j$ carefully. In particular we will constrain m so that it is possible to pick $s_j \in P_j$ such that $\sum_{j=1}^{m} \omega^{-rs_j} = 0$. For $s \in P_j$ we will approximate ω^{-rs} by ω^{-rs_j}. We skip the details of how good the approximation is.

We break up the sum over s via P_j.

$$\begin{aligned}
\hat{A}(r) &= \textstyle\sum_{s \in [n]} A(s)\omega^{-rs} \\
&= \textstyle\sum_{j=1}^{m} \sum_{s \in P_j} A(s)\omega^{-rs} \\
&\sim \textstyle\sum_{j=1}^{m} \sum_{s \in P_j} A(s)\omega^{-rs_j} \\
&= \textstyle\sum_{j=1}^{m} \omega^{-rs_j} \sum_{s \in P_j} A(s) \\
&= \textstyle\sum_{j=1}^{m} \omega^{-rs_j} \#(A \cap P_j) \\
&= \textstyle\sum_{j=1}^{m} \#(A \cap P_j)\omega^{-rs_j} \\
\alpha n \le |\hat{A}(r)| &= \textstyle|\sum_{j=1}^{m} \#(A \cap P_j)\omega^{-rs_j}|
\end{aligned}$$

We will not use ϵ. We intend to use Lemma 5.3; therefore we have the contraint $(\lambda - \frac{m-1}{m}(\lambda + \epsilon)) \ge 0$.

Assume, by way of contradiction, that $(\forall j)[|A \cap P_j| \le (\lambda + \epsilon)\frac{n}{m}$. Applying Lemma 5.3 we obtain

$$|\sum_{j=1}^{m} \#(A \cap P_j)\omega^{-rs_j}| \le \epsilon mn + (\lambda + \epsilon)\frac{n}{m}|\sum_{j=1}^{m} \omega^{-rs_j}| = \epsilon mn.$$

Hence we have

$\alpha n \le \epsilon mn$

$\alpha \le \epsilon m$.

In order to get a contradiction we pick ϵ and m such that $\alpha > \epsilon m$.

Having done that we now have that $(\exists j)[|A \cap P_j| \ge (\lambda + \epsilon)\frac{n}{m}]$.

We now list all of the constraints introduced and say how to satisfy them.

1. m is such that there exists $s_1 \in P_1, \ldots, s_m \in P_m$ such that $\sum_{j=1}^{m} \omega^{-rs_j} = 0$, and

2. $(\lambda - \frac{m-1}{m}(\lambda + \epsilon)) \ge 0$.

3. $\epsilon m < \alpha$.

First pick m to satisfy item 1. Then pick ϵ small enough to satisfy items 2,3. ∎

Lemma 5.5 *Let $A, B, C \subseteq [n]$. The number of 3-AP's $(x,y,z) \in A \times B \times C$ is bounded below by*

$$\frac{1}{2n}\sum_{r=1}^{n}\hat{A}(r)\hat{B}(-2r)\hat{C}(r) - O(n).$$

Proof:

The number of 3-AP's is bounded below by

$$\frac{1}{2n}\sum_{x,y,z\in[n]} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)} - O(n) =$$

We look at the inner sum.

$$\sum_{x,y,z\in[n]} A(x)B(y)C(z)\sum_{r=1}^{n}\omega^{-r(x-2y+z)} =$$

$$\sum_{r=1}^{n}\sum_{x,y,z\in[n]} A(x)\omega^{-rx}B(y)\omega^{2yr}C(z)\omega^{-rz} =$$

$$\sum_{r=1}^{n}\sum_{x\in[n]} A(x)\omega^{-rx}\sum_{y\in[n]} B(y)\omega^{2yr}\sum_{z\in \mathsf{Z}_r} C(z)\omega^{-rz} =$$

$$\sum_{r=1}^{n}\hat{A}(r)\hat{B}(-2r)\hat{C}(r).$$

The Lemma follows. ∎

6 Main Theorem

Theorem 6.1 *For all λ, $0 < \lambda < 1$, there exists $n_0 \in \mathsf{N}$ such that for all $n \geq n_0$, $sz(n) \leq \lambda n$.*

Proof:

Let $S(\lambda)$ be the statement

there exists n_0 such that, for all $n \geq n_0$, $sz(n) \leq \lambda n$.

It is a trivial exercise to show that $S(0.7)$ is true.

Let

$$C = \{\lambda : S(\lambda)\}.$$

C is closed upwards. Since $0.7 \in C$ we know $C \neq \emptyset$. Assume, by way of contradiction, that $C \neq (0,1)$. Then there exists $\lambda < \lambda_0$ such that $\lambda \notin C$ and $\lambda_0 \in C$. We can take $\lambda_0 - \lambda$ to be as small as we like. Let n_0 be such that $S(\lambda_0)$ is true via n_0. Let $n \geq n_0$ and let $A \subseteq [n]$ such that $\#(A) \geq \lambda n$ but A is 3-free.

Let $B = C = A \cap [n/3, 2n/3]$.

By Lemma 5.5 the number of 3-AP's of A is bounded below by

$$\frac{1}{2n}\sum_{r=1}^{n}\hat{A}(r)\hat{B}(-2r)\hat{C}(r) - O(n).$$

We will show that either this is positive or there exists a set $P \subseteq [n]$ that is an AP of length XXX and has density larger than λ. Hence P will have a 3-AP.

By Lemma 5.2 we have $\hat{A}(n) = \#(A)$, $\hat{B}(n) = \#(B)$, and $\hat{C}(n) = \#(C)$. Hence

$$\frac{1}{2n}\hat{A}(n)\hat{B}(n)\hat{C}(n) + \frac{1}{2n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r) - O(n) =$$

$$\frac{1}{2n}\#(A)\#(B)\#(C) + \frac{1}{2n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r) - O(n).$$

By Lemma 2.1 we can take $\#(B), \#(C) \geq n\lambda/4$. We already have $\#(A) \geq \lambda n$. This makes the lead term $\Omega(n^3)$; hence we can omit the $O(n)$ term. More precisely we have that the number of 3-AP's in A is bounded below by

$$\frac{\lambda^3 n^2}{32} + \frac{1}{2n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r)).$$

We are assuming that this quantity is ≤ 0.

$$\frac{\lambda^3 n^2}{32} + \frac{1}{2n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r)) < 0.$$

$$\frac{\lambda^3 n^2}{16} + \frac{1}{n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r)) < 0.$$

$$\frac{\lambda^3 n^2}{16} < -\frac{1}{n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r)).$$

Since the left hand side is positive we have

$$\begin{array}{rl} \frac{\lambda^3 n^2}{16} < & |\frac{1}{n}\sum_{r=1}^{n-1}\hat{A}(r)\hat{B}(-2r)\hat{C}(r)| \\ < & \frac{1}{n}(\max r\hat{A}(r))\sum_{r=1}^{n-1}|\hat{B}(-2r)||\hat{C}(r)| \end{array}$$

By the Cauchy Schwartz inequality we know that

$$\sum_{i=1}^{n-1}|\hat{B}(-2r)||\hat{C}(r)| \leq (\sum_{i=1}^{n-1}|\hat{B}(-2r)|^2)^{1/2})(\sum_{i=1}^{n-1}|\hat{C}(r)|^2)^{1/2}).$$

Hence

$$\frac{\lambda^3 n^2}{16} < |\frac{1}{n}\max_{1\leq r\leq n-1}|\hat{A}(r)|(\sum_{i=1}^{n-1}|\hat{B}(-2r)|^2)^{1/2})(\sum_{i=1}^{n-1}|\hat{C}(r)|^2)^{1/2}).$$

By Parsaval's inequality and the definition of B and C we have

$$\sum_{i=1}^{n-1} |\hat{B}(-2r)|^2)^{1/2} \leq n\#(B) = \frac{\lambda n^2}{3}$$

and

$$\sum_{i=1}^{n-1} |\hat{C}(r)|^2)^{1/2} \leq n\#(C) = \frac{\lambda n^2}{3}$$

Hence

$$\frac{\lambda^3 n^2}{16} < (\max_{1 \leq r \leq n-1} |\hat{A}(r)|)\frac{1}{n}\frac{\lambda n^2}{3} = (\max_{1 \leq r \leq n-1} |\hat{A}(r)|)\frac{\lambda n}{3}.$$

Therefore
$|\hat{A}(r) \geq \frac{3\lambda^2 n}{16}$. ∎

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