Second Order Statements True in $(R, +)$ but not $(Q, +)$
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1 Introduction

We give a statement in the second order language of $+$ that is true in $(\mathsf{R}, +)$ but false in $(\mathsf{Q}, +)$. We first give it in English.

There exists sets A, B such that both $(A, +)$ and $(B, +)$ are groups but $A \cap B = \{0\}$.
We now give this is as statement in second order $+$. We need some subformulas first.

1. Let $Z(x)$ be the formula
$$(\forall y)[x + y = y].$$
This says that $x = 0$. Note that $(\exists x)[Z(x)]$ is true in both R and Q and in both cases the x is 0.

2. Let $ZI(A, B)$ be the formula
$$(\forall x)[(x \in A \wedge x \in B) \implies Z(x)].$$
This says that the only element in $A \cup B$ is 0.

3. Let $CL(A)$ be the formula
$$(\forall x)(\forall y)[(x \in A \wedge y \in A) \implies x + y \in A].$$
This says that A is closed under addition.

4. Let $I(A)$ be the formula
$$(\forall x)(\exists y)[x \in A \implies Z(x + y)]$$
This says that A is closed under additive inverses.

5. Let $GR(A)$ be the formula
$$CL(A) \wedge I(A).$$
This says that A is a group.

Theorem 1.1 *Let ψ be the following sentence in the second order language of $+$.*
$$\psi = (\exists A)(\exists B)[GR(A) \wedge GR(B) \wedge ZI(A, B)].$$
Then

1. $(\mathsf{R}, +) \models \phi$,

2. $(\mathsf{Q}, +) \models \neg\phi$.

Proof:
We first show that the statement is true in R.

Let

$$A = \{q\pi \mid q \in \mathsf{Q}\}.$$

$$B = \mathsf{Q}.$$

Clearly both A and B are groups. One can easily show that if $x \in A \cap B$ then $x = 0$ (else $\pi \in \mathsf{Q}$).

We now show that the statement is false in Q. Assume, by way of contradiction, that the statement is true in Q. Let $\frac{p_1}{q_1} \in A \cap \mathsf{Q}^+$ and $\frac{p_2}{q_q} \in B \cap \mathsf{Q}^+$.

Since A is closed under addition, for all $n_1 \in \mathsf{N}$, $\frac{n_1 p_1}{q_1} \in A$. Since B is closed under addition, for all $n_2 \in \mathsf{N}$, $\frac{n_2 p_2}{q_2} \in B$. Let $n_1 = q_1 p_2$ and $n_2 = q_2 p_1$. This yields that $p_1 p_2 \in A$ and $p_1 p_2 \in B$. Hence there is a nonzero element in $A \cap B$. This is a contradiction. ∎

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