1 Introduction

We give a statement in the second order language of + that is true in (R,+) but false in (Q,+).

We first give it in English.

There exists sets $A, B$ such that both $(A, +)$ and $(B, +)$ are groups but $A \cap B = \{0\}$.

We now give this as statement in second order +. We need some subformulas first.

1. Let $Z(x)$ be the formula

   $$(\forall y)[x + y = y].$$

   This says that $x = 0$. Note that $(\exists x)[Z(x)]$ is true in both R and Q and in both cases the $x$ is 0.

2. Let $ZI(A, B)$ be the formula

   $$(\forall x)[(x \in A \land x \in B) \implies Z(x)].$$

   This says that the only element in $A \cup B$ is 0.

3. Let $CL(A)$ be the formula

   $$(\forall x)(\forall y)[(x \in A \land y \in A) \implies x + y \in A].$$

   This says that $A$ is closed under addition.

4. Let $I(A)$ be the formula

   $$(\forall x)(\exists y)[x \in A \implies Z(x + y)]$$

   This says that $A$ is closed under additive inverses.

5. Let $GR(A)$ be the formula

   $$CL(A) \land I(A).$$

   This says that $A$ is a group.

Theorem 1.1 Let $\psi$ be the following sentence in the second order language of +.

$$\psi = (\exists A)(\exists B)[GR(A) \land GR(B) \land ZI(A, B)].$$

Then
1. \((\mathbb{R}, +) \models \phi\),

2. \((\mathbb{Q}, +) \models \neg \phi\).

**Proof:**
We first show that the statement is true in \(\mathbb{R}\).

Let

\[ A = \{q\pi \mid q \in \mathbb{Q}\}. \]

\[ B = \mathbb{Q}. \]

Clearly both \(A\) and \(B\) are groups. One can easily show that if \(x \in A \cap B\) then \(x = 0\) (else \(\pi \in \mathbb{Q}\)).

We now show that the statement is false in \(\mathbb{Q}\). Assume, by way of contradiction, that the statement is true in \(\mathbb{Q}\). Let \(\frac{p_1}{q_1} \in A \cap \mathbb{Q}^+\) and \(\frac{p_2}{q_2} \in B \cap \mathbb{Q}^+\).

Since \(A\) is closed under addition, for all \(n_1 \in \mathbb{N}\), \(\frac{n_1p_1}{q_1} \in A\). Since \(B\) is closed under addition, for all \(n_2 \in \mathbb{N}\), \(\frac{n_2p_2}{q_2} \in B\). Let \(n_1 = q_1p_2\) and \(n_2 = q_2p_1\). This yields that \(p_1p_2 \in A\) and \(p_1p_2 \in B\). Hence there is a nonzero element in \(A \cap B\). This is a contradiction. \(\blacksquare\)

2  **Acknowledgments**

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