Van Der Waerden's Theorem: Exposition and Generalizations
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1 Introduction

In this paper we will present and prove van der Waerden's theorem and several generalizations of it.

Notation 1.1 If $m \in \mathsf{N}$ then $[m]$ is $\{1, \ldots, m\}$.

Definition 1.2 If $k \in \mathsf{N}$ then a k-AP is an arithmetic progression of length k. Henceforth we abbreviate "arithmetic progression' by AP and "arithmetic progression of length k" by k-AP.

The following statement is the original van der Waerden's Theorem:

Theorem 1.3 *[6] For every $k \geq 1$ and $c \geq 1$ there exists $W = W(k, c)$ such that for every c-coloring $COL : [W] \rightarrow [c]$ there exists a monochromatic k-AP. In other words there exists a, d such that*

- $a, a+d, a+2d, \ldots, a+(k-1)d \in [W]$, *and*
- $COL(a) = COL(a+d) = \cdots = COL(a+(k-1)d)$.

Note 1.4 When we speak of a c-coloring of $[W]$ we mean a mapping from $[W]$ to $[c]$. In particular, we always color with numbers.

The following is equivalent to van der Waerden's Theorem by a simple compactness argument.

Theorem 1.5 *For every $k \geq 1$ and $c \geq 1$ for every c-coloring $COL : \mathsf{Z} \rightarrow [c]$ there exists $a, d \in \mathsf{Z}$ such that*

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-1)d).$$

In Theorem 1.5 we can think of

$$a, a+d, \ldots, a+(k-1)d.$$

as

$$a, a+p_1(d), a+p_2(d), \ldots, a+p_{k-1}(d)$$

where $p_i(d) = id$. Why these functions? We ponder replacing p_i with other functions.

The following remarkable theorem was first proved by Bergelson and Leibman [1]. They proved it by first proving the polynomial version of the Hales-Jewitt Theorem [2] (see Section 4 for a statement and proof of the original Hales-Jewitt Theorem), from which Theorem 1.8 follows easily. Their proof of the polynomial version of the Hales-Jewitt Theorem used ergodic methods. A later proof by Walters [7] uses combinatorial techniques. Hence, putting all of this together, there is a combinatorial proof of Theorem 1.8. The purpose of this note is to put all of this together in a self-contained way.

Theorem 1.6 *For any natural number c and any polynomials* $p_1(x), \ldots, p_k(x) \in \mathsf{Z}[x]$ *such that* $(\forall i)[p_i(0) = 0]$*, for any c-coloring* $COL :: \mathsf{Z} \rightarrow [c]$ *there exists a* $a, d \in \mathsf{Z}$ *such that*

$$COL(a) = COL(a + p_1(d)) = COL(a + p_2(d)) = \cdots = COL(a + p_k(d)).$$

Note 1.7 This was proved for $k = 1$ by Furstenberg [3] and (independently) Sarkozy [5].

What if Z is replaced by another integral domain? Bergelson and Leibman [2] proved the following theorem.

Theorem 1.8 *Let* S *be any integral domain. Let* $c \in \mathsf{N}$*. Let* $p_1(x), \ldots, p_k(x) \in S[x]$ *be such that* $(\forall i)[p_i(0) = 0]$*. For any c-coloring COL of S there exists* $a, d \in S$ *such that*

$$COL(a) = COL(a + p_1(d)) = COL(a + p_2(d)) = \cdots = COL(p_k(d)).$$

Henceforth VDW means van der Waerden's Theorem, PVDW means the polynomial van der Warden's Theorem, HJ means the Hales-Jewitt Theorem, and PHJ means Polynomial Hales-Jewitt Theorem.

This exposition will contain the following:

1. The original proof of VDW. (Theorem 1.5)
2. The combinatorial proof of PVDW. (Theorem 1.6)
3. The original proof of HJ.
4. Shelah's proof of HJ which provides better bounds on the VDW numbers.
5. The combinatorial proof of PHJ.
6. The combinatorial proof of the generalized PVDW. (Theorem 1.8)

Since this is an exposition there will be more figures, examples, and detailed proofs than is common in a mathematics paper.

2 The Original Proof of Van Der Waerden's Theorem

We present the original proof of van der Waerden's Theorem. Our treatment is based on that of [4] but is more detailed.

2.1 Van Der Waerden's Theorem: Easy Cases

We present some easy cases of VDW theorem which we leave to the reader to prove.

1. If $k = 1$ and c is anything then $W(k, c) = 1$.
2. If $k = 2$ and c is anything then $W(k, c) = c + 1$. This is by the Pigeonhole Principle which we will be using over and over again.
3. If k is anything and $c = 1$ then $W(k, c) = k$.

2.2 The First Interesting Case: $W(3,2)$

We show that there exists a W such that any 2-coloring of $[W]$ has a monochromatic 3-AP.

Assume W is a multiple of 5, say $W = 5U$. View $[W]$ as being U blocks of 5 consecutive numbers each. We denote these blocks

$$B_1B_2\cdots B_U.$$

KEY insight: a 2-coloring of $[W]$ can be viewed as a 2^5-coloring of the blocks.

(This will be a recurring theme in later proofs: If $W = bU$ then we think of W as U blocks of b each, and we can think of a c-coloring of W as a c^b-coloring of the blocks.)

We leave the proofs of the following facts to the reader.

Fact 2.1 *Let* $c \in \mathsf{N}$.

1. *Let* B *be a block of* $2c+1$. *Let* $COL : B \rightarrow [c]$ *be a* c*-coloring of* B. *Then there exists* a, d *such that*
$$a, a+d, a+2d \in B$$
$$COL(a) = COL(a+d)$$
We make no comment on $COL(a+2d)$. *(See Picture)*

2. *Let* $W = b(2c^b+1)$. *We view* W *as* $2c^b+1$ *blocks of size* b *which we denote*
$$B_1B_2\cdots B_{2c^b+1}.$$
Let $COL : [W] \rightarrow [c]$ *be a* c*-coloring of* $[W]$ *and let* COL^* *be the induced* c^b*-coloring of the blocks. Then there exists* A, D *such that*
$$A, A+D, A+2D \in [2c^b+1]$$
$$COL^*(B_A) = COL^*(B_{A+D}).$$
We make no comment on $COL^*(B_{A+2D})$. *(See Picture)*

Theorem 2.2 *Let* $W = 5(2\times 32+1)$. *Let* $COL : [W] \rightarrow [2]$ *be a 2-coloring of* $[W]$. *Then there* a, d *such that such that*
$$a, d \in B_A$$
$$COL(a) = COL(a+d) = COL(a+2d).$$

Proof: We take the colors to be RED and BLUE.

View $[W]$ as being in $(2\times 32+1)$ blocks of 5. We denote the blocks

$$B_1B_2\cdots B_{2\times 32+1}.$$

Let COL^* be the induced 32-coloring of the blocks. By Fact 2.1.2 there exists A, D such that

$$A, A+D, A+2D \in [2\times 32+1]$$

$$COL^*(B_A) = COL^*(B_{A+D}).$$

By Fact 2.1.1 there exists a, d such that $a \in B_A$, $d \neq 0$, and $a + d \in B_A$.

$$COL(a) = COL(a + d).$$

We will assume the color is RED. Since $COL(a) = COL(a+d)$ and $COL^*(B_A) = COL^*(B_{A+D})$ we have

$$COL(a) = COL(a + d) = COL(a + D) = COL(a + d + D) = RED.$$

Since $COL^*(B_A) = COL^*(B_{A+D})$

$$COL(a + 2d) = COL(a + 2d + D).$$

We make no claim as to what $COL(a + 2d)$ is.
NEED PICTURE
There are two cases.

1. If $COL(a + 2d) = RED$ then $a, a + d, a + 2d$ are a RED 3-AP.
2. If $COL(a + 2d) = BLUE$ then $COL(a + 2d + D) = BLUE$.
   (a) If $COL(a+2d+2D) = BLUE$ then $a+2d, a+2d+D, a+2d+2D$ are a BLUE 3-AP.
   (b) If $COL(a + 2d + 2D) = RED$ then $a, a + d + D, a + 2d + 2D$ are a RED 3-AP.

∎

Note 2.3 The proof of Theorem 2.2 yields $W(3, 2) \leq 5 \times 65$. One can show by cases that $W(3, 2) = 9$. This can be done by hand and we urge the reader to try.

2.3 Van Der Waerden's Theorem for $k = 3$

Theorem 2.4 *For every $c \geq 1$ there exists $W = W(3, c)$ such that for every c-coloring COL of $[W]$ there exists a monochromatic 3-AP.*

The following lemma will easily yield Theorem 2.4.

Lemma 2.5 *If $c \geq 1$ and $1 \leq r \leq c$ then there exists $U = U(c, r)$ such that for any c-coloring COL of $[U]$ either*

1. *there exists a monochromatic 3-AP, or*
2. *there exists a $w \in [U]$ and a set $C \subseteq [c]$ such that*
   (a) *$|C| = r$,*
   (b) *$COL(w) \notin C$, and*
   (c) *if w is recolored with any color in C then there would be a monochromatic 3-AP.*

Proof: We do an induction on r, $1 \le r \le c$.
Base Case: We show that if $r = 1$ then $U(c, 1) = 2c+1$ suffices. Let COL be any c-coloring of $[2c + 1]$. By Fact 2.1.1 there exists a a, d such that

$$a, a + d, a + 2d \in [2c + 1]$$

$$COL(a) = COL(a + d).$$

If $COL(a + 2d) = COL(a)$ then $(a, a + d, a + 2d)$ form a monochromatic 3-AP and we are done. If not then let w be $a + 2d$ and let $C = \{COL(a)\}$. Clearly $|C| = 1$, $COL(w) \notin C$, and if w is recolored with any element of C then there will be a monochromatic 3-AP. Hence we are done.

r=2 Case: We do the $r = 2$ case even though it is not needed for the proof. We show that $U(c, 2) = U = 2(2c + 1)(c^{2c+1} + 1) + (2c + 1)$ suffices. Let COL be any c-coloring of $[U]$. Break $[U]$ into $2c^{2c+1} + 1$ consecutive blocks of size $2c + 1$ each. Let the blocks be

$$B_1 B_2 \cdots B_{2(c^{2c+1})+1}.$$

Let COL^* be the induced c^{2c+1}-coloring on the blocks. By Fact 2.1.2 there exists A, D such that

$$A, A + D, A + 2D \in [2c^{2c+1} + 1]$$

$$COL^*(B_A) = COL^*(B_{A+D}).$$

By Fact 2.1.1 there exists a, d such that

$$a, a + d, a + 2d \in B_A$$

$$COL(a) = COL(a + d).$$

We know the following

- $COL(a) = COL(a + d) = COL(a + D) = COL(a + d + D)$
- $COL(a + 2d) = COL(a + 2d + D)$.

There are several cases.

1. $COL(a + 2d) = COL(a)$. Then $a, a + d, a + 2d$ forms a monochromatic 3-AP. NEED PICTURE
2. $COL(a + 2d) = COL(a + 2d + 2D)$. Then $(a + 2d, a + 2d + D, a + 2d + 2D)$ forms a monochromatic 3-AP.
3. NEED PICTURE $COL(a + 2d) \ne COL(a)$ and $COL(a + 2d) \ne COL(a + 2d + 2D)$. Let $w = a + 2d + 2D$. Note that $w \in B_{A+2D}$ so, in particular, $w \in [U]$.

$$COL(a) = COL(a + d) = COL(a + D) = COL(a + d + D) = RED,$$

$$COL(a+2d) = COL(a+2d+D) = BLUE.$$

Let $C = \{RED, BLUE\}$. Clearly $|C| = 2$, $COL(w) \notin C$. If w is recolored RED then $(a, a+d+D, w)$ is a monochromatic 3-AP. If w is recolored BLUE then $(a+d, a+d+D, w)$ is a monochromatic 3-AP. NEED PICTURE

Induction Hypothesis: $U(c,r)$ exists.

Induction Step: We show that

$$U = U(c, r+1) = (2U(c,r)+1)c^{U(c,r)}$$

suffices. Let COL be any c-coloring of $[U]$. Break $[U]$ into $2c^{U(c,r)}+1$ consecutive blocks of size $U(c,r)$ each. Let the blocks be

$$B_1 B_2 \cdots B_{2c^{U(c,r)}+1}.$$

Let COL^* be the induced $c^{U(c,r)}$ coloring of the blocks. By Fact 2.1.2 there exists A, D such that

$$A, A+D, A+2D \in [2c^{U(c,r)}+1].$$

$$COL^*(B_A) = COL^*(B_{A+D}).$$

By the induction hypothesis applied to B_A we know that either B_A has a monochromatic 3-AP (in which case we are done, so we will ignore this case) or there exists $w_0 \in B_A$ and $C_0 \subseteq [c]$ such that the following hold.

1. $|C_0| = r$.
2. $COL(w_0) \notin C_0$.
3. If w_0 is recolored with any element of C_0 then there will be a monochromatic 3-AP in B_A.

By renumbering we assume $C_0 = [r]$ and w_0 is colored $r+1$. By the definition of C_0 we know that there exist $a_1, \ldots, a_r, d_1, \ldots, d_r$ such that the following hold.

0) $a_1, \ldots, a_r, a_1+d_1, \ldots, a_r+d_r \in B_A$,

1) $COL(a_1) = COL(a_1+d_1) = 1$, $w_0 = a_1+2d_1$, and $COL(w_0) \neq 1$.

2) $COL(a_2) = COL(a_2+d_2) = 2$, $w_0 = a_2+2d_2$, and $COL(w_0) \neq 2$.

$\vdots$

r) $COL(a_r) = COL(a_r+d_r) = r$, $w_0 = a_r+2d_r$, and $COL(w_0) \neq r$.

Since $COL^*(B_A) = COL^*(B_{A+D})$ we have the following.

1) $COL(a_1+D) = COL(a_1+D+d_1) = 1$, $w_0+D = a_1+D+2d_1$, and $COL(w_0+D) \neq 1$.

2) $COL(a_2+D) = COL(a_2+D+d_2) = 2$, $w_0+D = a_2+D+2d_2$, and $COL(w_0+D) \neq 2$.

$\vdots$

r) $COL(a_r+D) = COL(a_r+D+d_r) = r$, $w_0+D = a_r+D+2d_r$, and $COL(w_0+D) \neq r$.

Let $w = w_0 + 2D$. Since $w \in B_{A+2D}$, $w \in [U]$. There are several cases

Case 0: $COL(w) \in C$. Then we have a monochromatic 3-AP. Details left to the reader.

Case 1: $COL(w) = COL(w_0)$. Then

$$(w_0, w_0 + D, w) = (w_0, w_0 + D, w_0 + 2D)$$

form a monochromatic 3-AP.

Case 2: $COL(w) \neq COL(w_0)$. Let $C = C_0 \cup \{r + 1\} = [r + 1]$. Clearly $|C| = r + 1$ and $COL(w) \notin C$. If w is recolored to any of $1 \leq i \leq r$ then

$$(a_i, a_i + d_i + D, w) = (a_i, a_i + d_i + D, w_0 + 2D)(a_i, a_i + D + d_i, a_i + 2d_i + 2D)$$

form a monochromatic 3-AP. If w is recolored with $r + 1$ then $(w_0, w_0 + D, w_0 + 2D)$ forms a monochromatic 3-AP. Hence we have our desired number w and set C. ∎

Note 2.6 How fast does $U(c, r)$ grow?

1. $U(c, 1) = 2c + 1$.
2. $U(c, 2) = 2(2c + 1)c^{2c+1} = c^{O(c)}$.
3. $U(c, 3) = 2U(c, 2)c^{U(c,2)} = c^{O(c)}c^{c^{O(c)}} = c^{c^{O(c)}}$.

How to properly express this? Let $TOW(c, r)$ be $c^{c^{\cdots}}$ where the tower of exponents is r-high. We know that $U(c, r) \leq TOW(c, O(k))$. Hence $W(3, c) \leq TOW(c, O(c))$.

BILL- CHECK ON THE TOWER

Theorem 2.7 *If $c \geq 2$ then there exists $W = W(3, c)$ such that for any c-coloring COL of $[W]$ there exists a, d such that*

$$COL(a) = COL(a + d) = COL(a + 2d).$$

Moreover $W(3, c) \leq TOW(c, O(c))$.

Proof: Let $W(3, c) = U(c, c)$ where U was defined in Lemma 2.5. Let COL be any c-coloring of $[W]$. By Lemma 2.5 either there is a monochromatic 3-AP (so we are done) or there exists a $w \leq W$ and a set C such that $|C| = c$ such that $COL(w) \notin C$. This second case can't happen since COL is a c-coloring. Hence the first case happens so there is a monochromatic 3-AP. ∎

2.4 An Easy Lower Bound on $W(3,c)$

We now obtain a lower bound on $W(3,c)$. Much better lower bounds are known.

Theorem 2.8 *For c large $W(3,c) > 2c$.*

Proof: Let the colors be $[c]$. Use the coloring $112233\cdots cc$. ▮

2.5 A Proof of the Full VDW theorem

There are two parameters: k and c. Which one to do induction on? We will do induction on the ordered pair (k,c) under the following ordering.

$$(2,2) \prec (2,3) \prec (2,4) \prec \cdots (3,2) \prec (3,3) \prec (3,4) \prec \cdots \prec (4,2) \prec (4,3) \prec (4,4) \cdots$$

Formally the ordering is $(i,j) \prec (i',j')$ iff either $i < i'$ or $i = i'$ and $j < j'$.

Definition 2.9 An ordering is *well founded* if it has no infinite descending chains. These are precisely the orderings that one can do a proof by induction on.

The ordering $\prec$ is a well founded ordering. Note that even though there are plenty of $\cdots$'s, if you start anywhere in the ordering and try to go down for as far as you can, you will end up at $(2,2)$.

Example 2.10 Start at the element $(5,17)$. Let C be a decreasing chain that starts with $(5,17)$. We show that C is finite. We can assume that C begins

$$(5,17) \succeq (5,16) \succeq (5,15 \succeq \cdots \succeq (5,2)$$

The next point in C has to begin with a 4,3,2, or 1. We'll assume it begins with a 4. Say it is $(4,N)$. This is the key- it has to be $(4,N)$ where N is some finite number. After $N-2$ more steps in the chain you will have either $(4,2)$ or $(3,M)$ for some M. Continuing in this way eventually (after a FINITE number of steps) you will get to $(2,2)$.

We have already established the theorem for $(2,2),(2,3),\ldots,(3,2),(3,3),(3,4),\ldots$. The next case of interest is $(4,2)$. We will now proof the full VDW but note that the case of $(4,2)$ will depend on $(3,M)$ where M is very large.

Definition 2.11 If $A \subseteq \mathsf{N}$ and $D \in \mathsf{N}$ then

$$A + D = \{x + D \mid x \in A\}.$$

Usually A will be a finite contiguous subset of N.

We leave the proof of the following fact to the reader.

Fact 2.12 *Let $k \geq 3$. Assume that, for all c, $W(k-1,c)$ exists.*

1. *If COL is a c-coloring of $[2W(k-1,c)]$ then there exists a,d such that*

$$a, a+d, \ldots, a+(k-1)d \in [2W(k-1,c)]$$

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-2)d).$$

   We make no comment on $COL(a+(k-1)d)$.

2. *Let $b \in \mathsf{N}$. Let $W = b(2W(k-1,c^b))$.. We view W as $2W(k-1,c^b)$ blocks of size b. which we denote*

$$B_1 B_2 \cdots B_{2W(k-1,c^b)}.$$

   Let $COL : [W] \rightarrow [c]$ be a c-coloring of $[W]$ and let COL^* be the induced c^b-coloring of the blocks. Then there exists a block A and a number D such that

$$A, A+D, \ldots, A+(k-1)D \in [2W(k-1,c^b]$$

$$COL^*(B_A) = COL^*(B_{A+D}) = \cdots = COL^*(B_{A+(k-2)D}).$$

   We make no comment on $COL^*(B_{A+(k-1)D})$.

Theorem 2.13 *For every $k \geq 1$ and $c \geq 1$ there exists $W = W(k,c)$ such that for every c-coloring COL of $[W]$ there exists a monochromatic k-AP.*

We prove a lemma from which the theorem will follow easily.

Lemma 2.14 *Fix $c \geq 1$, $k \geq 1$. Assume that for all ordered pairs $(k',c') \prec (k,c)$, $W(k',c')$ exists. Let $1 \leq r \leq c$. Then there exists $U = U(k,c,r)$ such that for any c-coloring COL of $[U]$ either*

1. *there exists a monochromatic k-AP, or*
2. *there exists a $w \in [U]$ and a set $C \subseteq [c]$ such that*
   (a) *$|C| = r$,*
   (b) *$COL(w) \notin C$, and*
   (c) *if w is recolored with any color in C then there would be a monochromatic k-AP.*

Proof: We do an induction on r, $1 \leq r \leq c$.
Base Case: We show that if $r = 1$ then $U(1,k,c) = 2W(k-1,c)$ suffices. Let COL be any c-coloring of $[2W(k-1,c)]$. By Fact 2.12 there exists a,d such that

$$a, a+d, a+2d, \ldots, a+(k-1)d \in [2W(k-1,c)]$$

$$COL(a) = COL(a+d) = COL(a+2d) = \cdots = COL(a+(k-2)d).$$

If $COL(a+(k-1)d) = COL(a)$ then

$$(a, a+d, a+2d, \ldots, a+(k-1)d)$$

form a monochromatic k-AP and we are done. If not then let $w = a+(k-1)d$ and let $C = \{COL(a)\}$. Clearly $|C| = 1$, $COL(w) \notin C$. If w is recolored with any element of C then there will be a monochromatic k-AP. Hence we are done.

Induction Hypothesis: $U(k, c, r)$ exists.

Induction Step We show that $U(k, c, r+1)$ exists. Let $U = U(k, c, r+1) = U(k, c, r)2W(k-1, c^{U(k,c,r)})$. Let COL be any c-coloring of $[U]$. If COL has any monochromatic k-AP's then we are done. Hence we assume that there are none.

View $[U]$ as $2W(k-1, c^{U(k,c,r)})$ consecutive blocks of size $U(k, c, r)$ each. Let the blocks be

$$B_1 B_2 \cdots B_{2W(k-1,c^{U(k,c,r)})}.$$

View COL as a $c^{U(k-1,c,r)}$-coloring of the blocks. We call this coloring COL^*. By Fact 2.12.2 there exists a block A and a number D such that

$$A, A+D, \ldots, A+(k-1)D \in [2W(k-1, c^{U(k,c,r)})]$$

$$COL^*(B_A) = COL^*(B_{A+D}) = \cdots = COL^*(B_{A+(k-2)D}).$$

Let

$$E_1 = B_A, E_2 = B_{A+D}, \ldots, E_k = B_{A+(k-1)D}.$$

For every i, $1 \le i \le k-1$, E_i is of size $U(k, c, r)$. We apply the induction hypothesis to E_1. Since we are assuming that there are no monochromatic k-AP's, there exists w_0 and C_0 such that

1. $|C_0| = r$. We can renumber and assume $C_0 = [r]$.
2. w_0 is not colored any color in C_0. We can renumber and assume $COL(w_0) = r+1$.
3. If w_0 is recolored to anything in C_0 then there will be a monochromatic k-AP. Hence, for every $j \in C_0$ there exists a_j, d_j such that
   (a) $a_j, a_j + d_j, a_j + 2d_j, \ldots, a_j + (k-1)d_j \in E_1$,
   (b) $COL(a_j) = COL(a_j + d_j) = COL(a_j + 2d_j) = \cdots = COL(a_j + (k-2)d_j) = j$,
   (c) $w_0 = a_j + (k-1)d_j$

Since $COL^*(E_1) = \cdots = COL^*(E_{k-1})$ we have

$$COL(w_0) = COL(w_0 + D) = \cdots = COL(w_0 + (k-2)D)$$

and

for every $j \in C_0$,

$$COL(a_j) = COL(a_j + D) = COL(a_j + 2D) = \cdots = COL(a_j + (k-2)D) = j.$$

Combining this with

$$COL(a_j) = COL(a_j + d_j) = COL(a_j + 2d_j) = \cdots = COL(a_j + (k-2)d_j) = j$$

we get what we need which is

$$COL(a_j) = COL(a_j + d_j + D) = COL(a_j + 2(d_j + D)) = \cdots = COL(a_j + (k-2)(d_j + D)) = j.$$

If $COL(w_0 + (k-1)D) = COL(w_0)$ then there is a monochromatic k-AP:

$$w_0, w_0 + D, \ldots, w_0 + (k-1)D.$$

Hence we assume this is not the case.

Let w be $w_0 + (k-1)D$ and $C = C_0 \cup \{COL(w_0)\}$. Note that, for all $j \in C_0$,

$$w = w_0 + (k-1)D = a_j + (k-1)d_j + (k-1)D = a_j + (k-1)(d_j + D).$$

If we recolor w to any element in C then a monochromatic k-AP is formed:

1. Recolor w to some $j \in C_0$. Note that $w = a_j + (k-1)D$. Denote the recoloring by COL'. We have

$$\begin{aligned} COL'(a_j) = COL'(a_j + d_j + D) = \; & COL'(a_j + 2d_j + 2D) = \cdots = \\ & COL'(a_j + (k-2)(d_j + D)) = COL'(a_j + (k-1)(d_j + D)) = j. \end{aligned}$$

2. Recolor w to $COL'(w_0)$. Denote the recoloring by COL'. We have

$$\begin{aligned} COL'(w_0) = COL'(w_0 + D) = \; & COL'(w_0 + 2D) = \cdots = \\ & COL'(w_0 + (k-2)D) = COL'(w_0 + (k-1)D). \end{aligned}$$

■

3 The Polynomial VDW Theorem

We prove the theorem below which is known as the Polynomial VDW theorem. This theorem was first proved by Bergelson and Leibman [1] using ergodic methods, and later proved by Walters [7] later proved it using combinatorial techniques. We give the combinatorial proof.

Theorem 3.1 *For any natural number c and any polynomials $p_1(x), \ldots, p_k(x) \in \mathsf{Z}[x]$ such that $(\forall i)[p_i(0) = 0]$, for any c-coloring of Z, there exists $a, d \in \mathsf{Z}$ such that*

$$COL(a) = COL(a + p_1(d)) = COL(a + p_2(d)) = \cdots COL(a + p_k(d)).$$

We will give the combinatorial proof.

3.1 The case of $k = 1$ and $p_1(x) = x^2$

We will prove the following:

Theorem 3.2 *For all c there exists $W = W(c)$ such that for any c-coloring $COL : [W] \rightarrow [c]$ there exists a, d such that*

$$COL(a) = COL(a + d^2).$$

We prove the following lemma from which the theorem will easily follow.

Lemma 3.3 *Fix c. For all r there exists $Q = Q(c, r)$ such that for any c-coloring $COL : [Q] \rightarrow [c]$ one of the following holds.*

- *There exists a, d such that*

$$COL(a) = COL(a + d^2).$$

- *There exists $a, d_1, d_2, \ldots, d_r$ such that*

$$COL(a), COL(a + d_1^2), COL(a + d_2^2), \ldots, COL(a + d_r^2) \text{ are all different.}$$

Proof:

We proof this by induction on r.

Base Case: $r = 1$. Take $Q(1) = 2$. This is trivial.

Induction Hypothesis: There exists $Q = Q(c, r)$ such that for any c-coloring COL of $[Q]$ one of the following holds.

- There exists a, d such that

$$COL(a) = COL(a + d^2).$$

- There exists $a, d_1, d_2, \ldots, d_r$ such that

$$COL(a), COL(a + d_1^2), COL(a + d_2^2), \ldots, COL(a + d_r^2) \text{ are all different.}$$

Induction Step: Let $Q = Q(c, r+1) = (Q(c, r)W(2Q(c, r)+1, c^{Q(c,r)}))^2 + Q(c, r)W(2Q(c, r)+1, c^{Q(c,r)})$. Let COL be a c-coloring of $[Q]$. If $(\exists a, d)$ such that

$$COL(a) = COL(a + d^2)$$

then we are done; hence, we assume this is not the case.

We view $[Q]$ as one block of size $(Q(c, r)W(2Q(c, r)+1, c^{Q(c,r)}))^2$ (the big block) followed by $W(2Q(c, r) + 1, c^{Q(c,r)})$ blocks of size $Q(c, r)$ (the small blocks). We concentrate on the coloring of $[[Q]]$ just on the small blocks. Let COL^* be the $c^{Q(c,r)}$-coloring of the small blocks induced by COL. Since there are $W(2Q(c, r) + 1, c^{Q(c,r)})$ blocks, by Theorem 2.13 there exists a block A and a number D such that

$$COL^*(A) = COL^*(A + D) = COL^*(A + 2D) = \cdots = COL^*(A + 2Q(c, r)D).$$

NEED PICT[Q]RE

Since A is of size $Q(c,r)$ there exists $a, d_1, \ldots, d_r$ such that

$$a, a+d_1^2, \ldots, a+d_r^2 \in A$$

$$COL(a), COL(a+d_1^2), COL(a+d_2^2), COL(a+d_3^2), \ldots, COL(a+d_r^2) \text{ are all different .}$$

Since

$$COL^*(A) = COL^*(A+D) = COL^*(A+2D) = \cdots = COL^*(A+2Q(c,r)D).$$

We have

$$COL(a) = COL(a+D) = COL(a+2D) = \cdots = COL(a+2Q(c,r)D).$$

$$COL(a+d_1^2) = COL(a+d_1^2+D) = COL(a+d_1^2+2D) = \cdots = COL(a+d_1^2+2Q(c,r)D).$$

$$COL(a+d_2^2) = COL(a+d_2^2+D) = COL(a+d_2^2+2D) = \cdots = COL(a+d_2^2+2Q(c,r)D).$$

$$\vdots$$

$$COL(a+d_r^2) = COL(a+d_r^2+D) = COL(a+d_r^2+2D) = \cdots = COL(a+d_r^2+2Q(c,r)D).$$

Note that

$$(\forall i)[d_i \le |A| = Q(c,r)].$$

We will need this later.

NEED PICT[Q]RE

Note that $D \le Q(c,r)W(2Q(c,r)+1, c^{Q(c,r)})$. Since $[[Q]]$ has at least $(Q(c,r)W(2Q(c,r)+1, c^{Q(c,r)}))^2$ elements before a, the number $a - D^2$ is in $[[Q]]$.

Set $a' = a - D^2$. We need $r+1$ numbers that are a square away from a' and that are all different colors. The first element is easy: a, which differs from a' by D^2. Hence we set $e_1 = D$. We know that $COL(a') \ne COL(a'+e_1^2)$ since we are assuming we do not have a number and a square away being the same color

We want a e_2 such that

$$COL(a') \ne COL(a'+e_2^2)$$

$$COL(a'+d_1^2) \ne COL(a'+e_2^2)$$

Since

$$COL(a+d_1^2) = COL(a+d_1^2+D) = COL(a+d_1^2+2D) = \cdots = COL(a+d_1^2+2Q(c,r)D)$$

and that this color is different from $COL(a) = COL(a'+e_1^2)$, we seek a shift of $a+d_1^2$ by some multiple of D, say SD, such that $a+d_1^2+SD$ is a square away from $a' = a - D^2$. Note that the difference is

$$D^2 + SD + d_1^2.$$

Take $S = 2d_1$. Since $d_1 \le Q(c,r)$, $2d_1 \le 2Q(c,r)$, so the element FILL IN LATER- THE POINT IS THAT d_1 ISN"T THAT BIG This motivates setting $e_2 = (D + d_1)$. More generally, for $2 \le i \le r+1$, set $e_i = (D + d_{i-1})$. Summing up we have the following:

1. $a' = a - D^2$
2. $e_1 = D$
3. $(\forall i, 2 \le i \le r+1)[e_i = D + d_{i-1}]$

We show that a', $a'+e_1^2$, ..., $a'+d_{r+1}^2$ are all different colors.

$COL(a')$ differs from all of the colors, else we would have a number and its square the same color.

For notational convenience let $d_0 = 0$. For $l \le i \le r+1$

$$COL(a'+e_i^2) = COL(a - D^2 + (D+d_{i-1})^2) = COL(a + d_{i-1}^2 + 2Dd_{i-1}^2) = COL(a + d_{i-1}^2)$$

NEED PICT[Q]RE

Since for all $0 \le i < j \le r$

$$COL(a+d_i^2) \ne COL(a+d_j^2),$$

by the above equation, for all $1 \le i \le r+1$,

$$COL(a'+e_i^2) \ne COL(a'+e_j^2).$$

∎

4 The Hales-Jewitt Theorem

5 The Polynomial HJ Theorem

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