An existential fragment of second order logic

Eric Rosen*
Mathematische Grundlagen der Informatik
RWTH Aachen
erosen@informatik.rwth-aachen.de

January, 1997

1 Introduction

In this paper, we examine an existential fragment of second order logic. In particular, we are interested in analyzing the expressive power of this logic over the class of finite structures. Thus, this research is naturally connected to central areas of finite model theory and descriptive complexity theory. On the one hand, following Fagin's theorem that $\Sigma^1_1 = \mathrm{NP}$, fragments of Σ^1_1 have been studied extensively (e.g., see [2], [3], [8], [9]). A major goal of this line of research has been to develop logical tools to separate the expressive power of different langauges, in the hope that they will have applications to some of the important open questions in complexity theory. From a slightly different perspective, Kolaitis and Vardi ([15] and [16]) have investigated the existence of 0-1 laws for fragments of Σ^1_1 that are defined in terms of their first order quantifier prefix. On the other hand, 'existential' fragments of different languages have been studied in various contexts (e.g., [17], [24], [11]). The most well known such language is undoubtedly the database language Datalog. Here, we do not have a formal definition of 'existential logic', but the basic idea is that the syntax guarantees that every class definable in the logic is closed under extensions. Observe that in this sense, Σ^1_1, 'existential second order logic', is not an existential logic.

In connection with the subjects mentioned above, $\mathrm{SO}(\exists)$, the topic of this paper, seems to be interesting for a number of reasons. First, it is rather expressive. Already, $\Pi^1_1(\exists)$ strictly contains Datalog($\neg$) and can express coNP-complete problems. (It turns out that there is not a natural sublogic of $\mathrm{SO}(\exists)$ that defines problems in NP. $\Sigma^1_1(\exists)$ is equivalent to the existential fragment of first order logic, which is extremely weak.) More generally, for each $n \in \omega$, there

*This research was partially supported by a postdoctoral fellowship from the Israel Council for Higher Education and by a research grant of the German-Israeli Foundation (GIF).

is a complete problem for Π^p_{2n+1}, in the polynomial hierarchy, that is definable in $\Pi^1_{2n+1}(\exists)$. Second, we introduce a technique using a result from Ramsey theory (see [22]) for proving non-definability results. In particular, we show that there is a class $\mathcal{C}$ that is in NP and closed under extensions that is not definable in full SO($\exists$). While the use of Ramsey theory to prove non-definability is not entirely new (e.g., see [14]), the context here is somewhat different. Our use of the theorem of Nešetřil and Rödl, though, was directly inspired by an idea of Kolaitis and Vardi [15]. Finally, we believe that interesting new questions arise in connection with SO($\exists$). For example, apparently difficult open problems regarding separations between fragments of second order logic (see [8]) can be reformulated in terms of SO($\exists$), and may be more tractable in this context.

The dual logic SO($\forall$) was studied by Mal′tsev [20] and also by Kreisel and Krivine [18], who proved independently that every SO($\forall$) sentence is equivalent to a set of universal FO sentences. More recently, SO($\exists$) has also been investigated in the context of finite model theory by Lacoste [19].

In Section 2, we provide definitions and establish some basic properties of SO($\exists$). In order to motivate the study of this language, we also present a number of examples of properties that are definable in it. Section 3 contains a number of general and basic model theoretic results about SO($\exists$). First, we establish the decidability of its satisfaction problem using a theorem due to Nešetřil and Rödl. We also discuss some consequences of the fact that SO($\exists$) has the *finite submodel property*, which is an immediate corollary of the result of Mal′tsev and of Kreisel and Krivine mentioned above. In particular, this property generalizes the compactness principle from Ramsey theory (see [10]) and implies that the language has some nice model theoretic properties, e.g. the downward Löwenheim-Skolem property.

Sections 4 and 5 examine questions regarding definability in SO($\exists$) that are more in the spirit of finite model theory. In the Section 4, we prove some facts about the expressive power of SO($\exists$) over finite structures. In particular, we show how the Nešetřil and Rödl theorem can be used to obtain non-definability results. In the next section, we consider the finite variable fragments of SO($\exists$), and use the same machinery to prove that they form a strict hierarchy. We also prove that for every purely relational finite model A whose signature contains a relation symbol of arity ≥ 4, the property of containing a submodel isomorphic to A is not expressible by any SO($\exists$) sentence containing less than $|A|$ (reusable) first order variables.

It is perhaps worth noting that in the proofs of the non-definability results that use the Nešetřil and Rödl theorem, we always (implicitly or explicitly) make use of a built in order on the universe, so that the arguments simultaneously prove the results over the class of ordered structures. In this sense, the situation here for SO($\exists$) contrasts sharply with the case of m.Σ^1_1, where it is much more difficult to obtain non-definability results over ordered structures (see [25]). On the other hand, the proofs do not carry over to classes with a built in successor relation (which is not definable from a linear order in SO($\exists$)).

Acknowledgements

I am grateful to Prof. Janos Makowsky for many valuable discussions on the subject of this paper, and to Prof. Jörg Flum for bringing the work of Kreisel and Krivine to my attention. I would also like to thank Joel David Hamkins for allowing me to include Theorem 4, which is due to him.

2 Definitions and basic results

We first recall some standard concepts and fix our notation. We use FO($\exists$) [FO($\forall$)] to refer to the set of existential [universal] first order formulas, and SO for second order logic. $sig(A)$ and $sig(\varphi)$ denote the signature of a structure and a formula, respectively. Given a signature σ, the *arity* of σ is the maximum arity of any relation $R \in \sigma$. Unless noted otherwise, signatures are always finite and relational. Given a structure A and a signature ρ, $sig(A) \cap \rho = \emptyset$, we use (A, ρ) or (A, ρ^A) to denote an expansion of A that interprets each symbol from ρ in A. Going in the other direction, if A is a σ model and $\tau \subseteq \sigma$, then $A|\tau$ denotes the τ-reduct of A.

We sometimes write $A_<$, etc., to indicate a linear order on the universe of A that is not part of the logical vocabulary. In this case, we will say that $A_<$ is an *ordered model* or an *ordered expansion* of A. We will sometimes also call models of the form $(A_<, \rho)$ ordered expansions of A. In some contexts, it will be more convenient to include the symbol $<$ in our signature σ and to restrict our attention to σ models with a *built in order*, that is, the class $\mathcal{O}_\sigma = \{A \mid A| < \text{ is a linear order}\}$. For A a model, R^A the interpretation of a k-ary relation symbol in A, and $B \subseteq A$, we write $R^A|B$ for the restriction of R^A to the universe of B, that is, $R^A|B = R^A \cap B^k$. In particular, if (A, R^A) is an R expansion of A, and $B \subseteq A$, then $(B, R^A|B)$ is a submodel of (A, R^A).

The following game provides a useful characterization of the satisfaction of a formula in a model.

Definition 1 *Let φ be a prenexed second order sentence, $\varphi = \mathcal{Q}_1 \ldots \mathcal{Q}_n \theta(\overline{x})$, $\mathcal{Q}_i$ a first or second order quantifier, θ quantifier free. The* φ-game *on A is an n round game played by two players, $\exists$ and $\forall$, on the universe of the structure. In each round i, $\exists$ plays iff $\mathcal{Q}_i$ is existential. If $\mathcal{Q}_i$ binds a relation symbol R then the appropriate player adds an interpretation of R to A. If $\mathcal{Q}_i$ binds a first order variable x_i, then the player plays a constant c_i on some element of A. The* position *of the φ-game after m rounds is the expansion of A by the relevant relations and constants.*

$\exists$ wins the game after round n iff the resulting expansion of A satisfies $\theta(\overline{c})$.

Proposition 1 *For all second order sentences φ and all models A, $\exists$ has a winning strategy in the φ-game on A iff $A \models \varphi$.*

We now introduce the fragment of second order logic that is the subject of this paper.

Definition 2 *Let* $\mathrm{SO}(\exists)$ *be the set of SO sentences, in prenex normal form, whose quantifier prefix is an arbitrary string of SO quantifiers followed by a string of existential first order (FO) quantifiers. Let* $\Pi^1_n(\exists)[\Sigma^1_n(\exists)]$ *be the fragment of* $\mathrm{SO}(\exists)$ *that consists of those sentences whose second order quantifer prefix consists of* n *alternating blocks of quantifiers, beginning with* $\forall[\exists]$*. Monadic* $\mathrm{SO}(\exists)$*, denoted* $m.\mathrm{SO}(\exists)$*, is the language in which all quantified SO relations are unary. Other languages, such as* $\mathrm{SO}(\forall)$ *and* $m.\Pi^1_n(\exists)$*, are defined in the obvious manner.*

The next proposition states the fundamental property of all classes defined by $\mathrm{SO}(\exists)$ sentences.

Proposition 2 *For all* $\varphi \in \mathrm{SO}(\exists)$*,* $\mathrm{Mod}(\varphi)$ *is closed under extensions.*

In particular, $\mathrm{SO}(\exists)$ does not have the full expressive power of first order logic. The following examples show that the expressive power of $\Pi^1_1(\exists)$, and hence also of $\mathrm{SO}(\exists)$, is incomparable with that of FO. Observe that $\Sigma^1_1(\exists)$ is equivalent to $\mathrm{FO}(\exists)$, the existential fragment of first order logic.

Example 1 *Over the signature* $\sigma = \{s, t, Exy\}$*, let* $\mathcal{C}$ *be the class of* (s,t)*-connected graphs, that is,* $\mathcal{C} = \{A \mid \text{there is a path from } s \text{ to } t\}$*. Then* $\mathcal{C}$ *is defined by the following sentence.*

$$\varphi = \forall Rxy \exists xyz((Exy \wedge \neg Rxy) \vee (Rxy \wedge Ryz \wedge \neg Rxz) \vee Rst)$$

On any graph A, φ *says that for every relation* R^A*, if* $E^A \subseteq R^A$ *and* R^A *is transitively closed, then* $A \models Rst$*.*

Generalizing this idea yields the following proposition, due to Blass and Gurevich [4]. (For more information on Datalog, see also [17] or [1].)

Proposition 3 *Every class* $\mathcal{C}$ *that is definable in Datalog*$(\neg)$ *is defined by a sentence in* $\Pi^1_1(\exists)$*.*

In this paper, a *graph* is always a (possibly infinite) undirected loop-free graph.

Example 2 *For each* n*, the class of graphs that are not* n*-colorable is definable in* $\Pi^1_1(\exists)$*. For example,*

$$\forall P_1 x P_2 x P_3 x (\exists x \neg (P_1 x \vee P_2 x \vee P_3 x) \vee \exists xy \bigvee_{i \leq 3} (Exy \wedge P_i x \wedge P_i y))$$

defines the class of non 3-colorable graphs.

Dawar [7] proved that, over the class of finite models, non 3-colorability is not even definable in $L^{\omega}_{\infty\omega}$, infinitary finite variable logic, which is strictly more expressive than least fixed point logic. On the other hand, since $L^{\omega}_{\infty\omega}(\exists)$, the existential fragment of $L^{\omega}_{\infty\omega}$, can express non-recursive queries, SO($\exists$) and $L^{\omega}_{\infty\omega}(\exists)$ have incomparable expressive power.

In fact, many combinatorial properties can be expressed in SO($\exists$), or even $\Pi^1_1(\exists)$. We provide two more examples. Given graphs F, G, H, let $F \to (G, H)$ mean that if the edges of F are colored 'red' and 'blue', then there must be either an induced subgraph isomorphic to G colored red or an induced sugbraph isomorphic to H colored blue. It is known, for example, that if G and H are triangles, then the class of graphs F such $F \to (G, H)$ is coNP-complete (e.g. see [5]).

Example 3 *Let G and H be fixed graphs of cardinality m and n respectively. Then $\{F \mid F \to (G, H)\}$ is definable over the class of graphs by the following $\Pi^1_1(\exists)$ sentence.*

$$\forall Rxy \exists x_0 \ldots x_{m-1} \exists y_0 \ldots y_{n-1}((\theta_G(\overline{x}) \wedge \bigwedge_{i<j\leq m-1} (Ex_ix_j \to (Rx_ix_j \vee Rx_jx_i)))$$

$$\vee(\theta_H(\overline{y}) \wedge \bigwedge_{i<j\leq n-1} (Ey_iy_j \to \neg(Ry_iy_j \vee Ry_jy_i))))$$

where $\theta_G(\overline{x})$ is a quantifier free formula such that for all graphs $A, A \models \theta_G(\overline{a})$ iff the induced subgraph on $\overline{a}$ is isomorphic to G (likewise for $\theta_H(\overline{y})$ and H).

The following proposition illustrates a connection between SO($\exists$) and complexity theory. A version of Proposition 4 was discovered independently by Lacoste [19].

Proposition 4 *For all n, there is a $\varphi \in m.\Pi^1_{2n+1}(\exists)[m.\Sigma^1_{2(n+1)}(\exists)]$ such that $\mathcal{C} = Mod(\varphi)$ is Π^p_{2n+1}-complete $[\Sigma^p_{2(n+1)}$-complete$]$.*

Proof. We show that the satisfaction problem for *quantified Boolean formulas* with fixed number of quantifier alternations can be expressed in m.SO($\exists$). Recall that a quantified Boolean formula is a sentence of the form $(Q_1x_1)\ldots(Q_jx_j)\alpha$, where α is a propositional formula and each quantifier Q_i binds a propositional variable x_i. Let $QBF_{k,\exists}[QBF_{k,\forall}]$ be the set of true QBF formulas whose quantifer prefix consists of k alternating quantifier blocks beginning with $\exists$ $[\forall]$ (see [13]). It is well-known that $QBF_{k,\exists}$ $[QBF_{k,\exists}]$ is a complete problem for the class Σ^p_k $[\Pi^p_k]$ in the polynomial hierarchy (PH).

First we show how we code formulas as models. Fix a number $n \in \omega$, where n will be the (maximum) number of quantifier blocks in the formulas that are represented. Let $\sigma_n = \{Exy, c, L_\wedge, L_\vee, L_\neg, Y_1, \ldots, Y_n\}$, Exy a binary relation, c a constant, and all other symbols unary relations. A formula φ is coded as a model A^φ whose universe consists of two disjoint sets, T^φ and V^φ. T^φ represents the formula as a tree, while elements of V^φ correspond to quantified variables in the formula. Let $V^\varphi = \{x_j \mid x_j \text{ occurs in } \varphi\}$ and interpret Y_l as the set of elements x_j in V^φ such that x_j occurs in the l^{th} quantifier block. (Thus, the interpretations of the predicates Y_l partition the set V^φ.) The quantifier free matrix of φ is represented in the obvious way as a tree. Each element of T^φ corresponds to a subformula, and for any two such elements a_1, a_2, there is an edge Ea_1a_2 iff a_2 represents a maximal subformula of a_1. In particular, the leaves of the tree correspond to variable occurences and the root represents the entire formula. The constant names the root, and each non-leaf is labeled by exactly one of the predicates $L_\wedge, L_\vee, L_\neg$, in the obvious way. Finally for each element $x_i \in V^\varphi$ and each leaf $a \in T^\varphi$, there is an edge Ex_ia iff a represents an occurence of the variable x_i. (Thus, Exy defines a tree only when restricted to the set T^φ.)

We make the following two observations. First, since we do not represent the order of variables within a quantifier block, nor the order of subformulas, there are distinct formulas φ and θ such that $A^\varphi \cong A^\theta$. But it is easy to see that whenever this happens, then φ and θ are logically equivalent. Second, it is clear that for any φ the model A^φ can be constructed in polynomial time, which gives us the necessary reduction. Thus it only remains to show that for each odd n [even n], there is a formula $\psi \in m.\Pi^1_n(\exists)$ [$m.\Sigma^1_n(\exists)$] such that for each quantified Boolean formula φ with quantifier prefix in Π_n [Σ_n], $A^\varphi \models \psi$ iff φ is in $\mathrm{QBF}_{n,\forall}$ [$\mathrm{QBF}_{n,\exists}$].

For n odd, let ψ be the following sentence.

$$\forall S_1 \exists S_2 \forall S_3 \ldots \forall S_n \forall T (\bigvee_{i \leq n} \exists xy(Y_i x \wedge Exy \wedge (S_i x \leftrightarrow \neg Ty))$$

$$\vee \exists xyz(P_\wedge x \wedge Exy \wedge Exz \wedge (Tx \leftrightarrow \neg(Ty \wedge Tz)))$$

$$\vee \exists xyz(P_\vee \wedge Exy \wedge Exz \wedge (Tx \leftrightarrow \neg(Ty \vee Tz)))$$

$$\vee \exists xy(P_\neg x \wedge Exy \wedge (Tx \leftrightarrow Ty)) \vee Tc)$$

Intuitively, we use the relations S_i to assign truth values to the variables in Y_i and we use the relation T ('true') to evaluate the truth of each subformula under the assignment determined by the S_i from the 'bottom-up', starting at the variable occurences. One of the first four disjuncts is true iff T is an incorrect evaluation, so the whole sentence says that either T is incorrect or Tc, i.e. the formula is true. ■

One might wonder whether the expressive power of $\mathrm{SO}(\exists)$ [$\Pi^1_n(\exists)$] would be increased by allowing existential FO quantifiers to occur anywhere in the quantifier prefix. It turns out that this change would not make a difference.

It is well-known that any SO sentence φ can be transformed into another such sentence in prefix normal form such that every SO quantifier is outside of every FO quantifier. For the sake of completeness, we show that if φ is already in prefix normal form, then this can done without increasing either the number of SO quantifiers or alternations, and without introducing any new FO universal quantifiers. It clearly suffices to show that any sentence of the form $\theta = \exists x \mathcal{Q} S \psi$, where $\mathcal{Q}$ is a second order quantifier, is equivalent to a sentence of the form $\mathcal{Q} S' \exists x \psi'$, where ψ and ψ' have identical quantifier prefixes. If $\mathcal{Q}$ is $\exists$ then θ is obviously equivalent to $\exists S \exists x \psi$. On the other hand, if $\mathcal{Q}$ is $\forall$, then it is easy to verify that θ is equivalent to $\forall S' \exists x \psi'$, where $arity(S') = arity(S) + 1$ and ψ' is obtained by replacing each occurence of an atomic formula of the form $S(t_1, \ldots, t_n)$ by the formula $S'(t_1, \ldots, t_n, x)$.

3 Decidability

In this section, we establish some basic facts about SO($\exists$). First, we show that there is a decision procedure that determines whether any sentence in SO($\exists$) is satisfiable. In particular, we use a generalization of Ramsey's theorem first proved in Nešetřil and Rödl [21] to show that there is a recursive function $f(x)$ such that for all $\varphi \in$ SO($\exists$), φ is satisfiable iff it has a model of cardinality $\leq f(\varphi)$. [Observe that validity of SO($\exists$) sentences is trivially decidable, since each such φ is valid iff it is true in every model of size $= 1$.] We then discuss some consequences of the theorem, due to Mal′tsev and to Kreisel and Krivine, that SO($\exists$) has the *finite submodel property*, that is, for all $\varphi \in$ SO($\exists$) and all A, if $A \models \varphi$, then there is a finite submodel $B \subseteq A$ such that $B \models \varphi$. This result generalizes the compactness principle from Ramsey theory (see [10]) and can be proved in much the same way (using the Tychonoff theorem from topology), though their proofs are purely model theoretic. [The compactness principle states that if an infinite graph G is not r-colorable, then there is a finite subgraph $H \subseteq G$ that is not r-colorable.]

3.1 Ramsey theoretic background

In this section, we present the theorem of Nešetřil and Rödl which is the main technical tool used in the proofs of Theorem 2, 5, and 6. We first introduce some model theoretic preliminaries.

For each $j, j \geq 0$ a *(complete atomic) j-σ-type* is the conjunction of a maximally consistent set of basic (i.e. atomic or negated atomic) formulas with logical symbols from σ and variables from $x_1, \ldots, x_j$. We restrict our attention to 'injective' types, that is, those types that contain conjuncts $x_i \neq x_{i'}$, for all $i < i' \leq j$, For each j, let Θ_j^σ be the (finite) set of j-σ-types. For each (injective) j-tuple $\overline{a} \subseteq A$, we define the *(complete atomic) j-σ-type*, $tp(\overline{a})$ of $\overline{a}$ to be the unique $\tau(\overline{x}) \in \Theta_j^\sigma$ such that $A \models \tau[\overline{a}]$. When $A_<$ is an ordered model, we will only consider (types of) 'monotone' tuples, that is, those $\overline{a} = (a_1, \ldots, a_j)$ such that for all $i < i' \leq j$, $a_i < a_{i'}$. This allows us to associate a unique type with each *set* of elements.

If R is a k-ary relation symbol and $\overline{x}$ is a k-tuple of variables, we say that $R\overline{x}$ is an *R formula* (or an i-R formula) if $\overline{x} = \{x_1, \ldots, x_i\}$, for some $i \leq k$. For example, $Rx_2x_3x_1$ and $Rx_1x_2x_1$ are R formulas, but $Rx_2x_3x_4$ is not. Observe that for each relation symbol R, there are finitely many R formulas. For any signature σ, let Γ^σ denote the the union of all R_m formulas, $R_m \in \sigma$.

To state the result of Nešetřil and Rödl, we need the following concepts.

Definition 3 *1. A* kind *is a sequence $\Delta = (\delta_1, \ldots, \delta_t), \delta_s \in \mathbf{N}$, the set of natural numbers.*

2. A set system *of kind Δ is a pair $(X, \mathcal{M})$, with X an ordered finite set and $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_t)$ a sequence of sets such that for all $s \leq t$, if $Y \in \mathcal{M}_s$, then $Y \subseteq X$ and $|Y| = \delta_s$. We assume that $X \subseteq \mathbf{N}$ and inherits the natural ordering.*

3. $(X, \mathcal{M})$ is irreducible *iff for all $x, y \in X$, there is $M \in \bigcup \mathcal{M}$ such that $\{x, y\} \subseteq M$.*

4. $B = (X^B, \mathcal{M}^B)$ is a weak subsystem *of $A = (X^A, \mathcal{M}^A)$, written $B \subseteq_w A$, iff $X^B \subseteq X^A$ and for all $s \leq t$, $\mathcal{M}_s^B \subseteq \mathcal{M}_s^A$. B is an* induced subsystem *of A, $B \subseteq A$, iff $X^B \subseteq X^A$ and for all $s \leq t$, $\mathcal{M}_s^B = \mathcal{M}_s^A \cap [X^B]^{\delta_s}$.*

5. Soc(Δ) *is the set of all set systems of kind Δ. Let $\mathcal{A}$ be a set of irreducible set systems of kind Δ, closed under isomorphism. Then,*

$$Soc(\Delta, \mathcal{A}) = \{A \mid A \in Soc(\Delta) \text{ and for all } B \subseteq_w A \text{ then } B \notin \mathcal{A}\}$$

6. A system of colors *of arity n is a sequence $\chi = (c_1, \ldots, c_n), c_m \in \mathbf{N} - \{0\}$. Each $c \in \mathbf{N}$ is identified with its set of predecessors, that is, $c = \{0, 1, \ldots, c-1\}$. A* χ-coloring *of $(X, \mathcal{M})$ is a function $f_\chi : [X]^{\leq n} \longrightarrow \bigcup_{m \leq n} c_m$ such that for all $Y \in [X]^m, f_\chi(Y) \in c_m$. A χ-coloring f_χ of $(X, \mathcal{M})$ is* homogeneous *iff for all m and all $Y, Z \subseteq [X]^m$, if $(Y, \mathcal{M}^Y) \cong (Z, \mathcal{M}^Z)$, then $f_\chi(Y) = f_\chi(Z)$.*

7. For $B, C \in Soc(\Delta)$, and χ a system of colors, we write $C \hookrightarrow^\chi B$ iff every χ-coloring f_χ of C there is a $B' \subseteq C$, $B' \cong B$, that is colored homogeneously by $f_\chi | B'$.

Theorem 1 (Nešetřil and Rödl [21]) *Let Δ be a kind, χ a system of colors, and $\mathcal{A}$ a set of irreducible set systems. Then for all $B \in Soc(\Delta, \mathcal{A})$, there is a $C \in Soc(\Delta, \mathcal{A})$ such that $C \hookrightarrow^\chi B$.*

There is a straightforward way to view any ordered model as a set system. Let σ be a relational signature and recall that Γ^σ is the union of all R_m formulas, $R_m \in \sigma$. Choose $\Delta = (\delta_1, \ldots, \delta_{|\Gamma^\sigma|})$ to be any kind such that there is a bijection $h(x)$ from Γ^σ to $\{1, \ldots, |\Gamma^\sigma|\}$ such that for each i-R-formula $\theta \in \Gamma^\sigma, \delta_{h(\theta)} = i$. Then each $A_<$ with signature σ corresponds to the unique set system of kind Δ over the same (ordered) universe, such that for each (monotone) i-tuple $\overline{a}$, and each i-R formula $\theta(\overline{x}) \in \Gamma^\sigma$, $A_< \models \theta[\overline{a}]$ iff $\overline{a} \in \mathcal{M}_{h(\theta)}$. In fact, it is clear that this correspondence induces a natural bijection between ordered σ models and set systems of kind Δ. Likewise, expansions of ordered models can be treated as colorings of set systems. If ρ is a signature of arity k, then it corresponds to a system of colors of arity k, $(c_1, \ldots, c_k)$, such that for each $i \leq k, c_i$ is the number of atomic formulas over ρ whose free variables are exactly $\{x_1, \ldots, x_i\}$.

By (implicitly) relying on these correspondences, we can apply the terminology of set systems to ordered models. For example, we will often say that an expansion $(A_<, \overline{R}^A)$ is a homogeneous $\overline{R}$-coloring of $A_<$. Also, observe that an ordered σ model $A_<$ is irreducible iff for all $a_1, a_2 \in A$, there is a k-ary relation $R \in \sigma$ and a k-tuple $\overline{b}$ in A such that $a_1, a_2 \in \overline{b}$ and $R\overline{b}$. Below we will also say that an arbitrary (not ordered) model A is irreducible if it satisfies this condition. On the other hand, when we are considering models in $\mathcal{O}_\sigma$, with a built in order, we define a model to be irreducible iff every pair of elements is 'connected' by some relation $R \in \sigma \setminus \{<\}$. (Otherwise, every $A \in \mathcal{O}_\sigma$ would be irreducible.) Finally, if $\mathcal{A}$ is a set of irreducible σ models, then $Soc(\mathcal{A})$ denotes the set of [ordered] σ models that do not contain any $A \in \mathcal{A}$ as a weak submodel.

The following definition from Kolaitis and Vardi [15] plays a crucial role in our proofs.

Definition 4 *Let $A_<$ be an ordered model. For $k \in \omega$, we say that $A_<$ is k-$\overline{P}$-*rich *iff for every complete atomic k-$\overline{P}$-type $\tau(x_1, \ldots, x_k)$, there is a monotone k-tuple $\overline{a} = (a_1, \ldots, a_k) \subseteq A$ such that $A \models \tau[\overline{a}]$.*

Since there are only finitely many k-$\overline{P}$-types, it is obvious that for all k and $\overline{P}$, there are (finite) k-$\overline{P}$-rich ordered models.

3.2 Decidability

Theorem 2 *The satisfiability problem for the language* SO($\exists$) *is decidable. Specifically, there is a recursive function $f(x)$ such that for all* SO($\exists$) *sentences φ, if φ is satisfiable, then it has a model of cardinality $\leq f(\varphi)$.*

The theorem follows immediately from the following proposition.

Proposition 5 *For every φ in* SO($\forall$)*, there is a finite model A^φ, which can be found effectively, such that φ is valid iff $A^\varphi \models \varphi$.*

Proof of proposition. Let $\varphi = \exists\overline{R}_1\forall\overline{S}_1\exists\overline{R}_2 \ldots \exists\overline{R}_{n+1}\forall\overline{x}\theta(\overline{P}, \overline{R}, \overline{S}, \overline{x})$, $\varphi \in \Sigma^1_{2n+1}(\forall)$, where θ is quantifier free, $\overline{P}$ is the (purely relational) signature of

φ, $\overline{R} = \bigcup_{i \leq n+1} \overline{R}_i$ and $\overline{S} = \bigcup_{i \leq n} \overline{S}_i$. k is $length(\overline{x})$, and we assume that the arity of every relation is $\leq k$. [The argument generalizes to allow the addition of constants to the signature.] Let $\sigma_0 = \overline{P}$, and, for $1 \leq m \leq n$, $\sigma_m = \overline{P} \cup \bigcup_{l \leq m} \overline{S}_l$. By Definition 1, φ is valid iff $\exists$ wins the φ-game on every structure. The basic idea is to find a finite structure A^φ such that if $\exists$ has a winning strategy on A^φ, then this proves that he can win the φ-game on every structure. The existence of A^φ will be proved using the result of Nešetřil and Rödl. In essence, we will show that if $\exists$ wins on A^φ, then he has a 'uniform' winning strategy that can be used on every model.

Definition 5 *An ordered model $B_< = (A_<, \overline{R}, \overline{S})$ is* stepwise $\overline{R}, \overline{S}$ homogeneous *iff it is k-σ_n-rich and for all $m, 0 \leq m \leq n$, $B^m_< = B_< | \sigma_m$ is homogeneously $\overline{R}_{m+1}$-colored in the expansion $B_<$. We will also say that a model A with signature σ_0 is stepwise $\overline{R}, \overline{S}$ homogeneous if there is an ordered expansion $B_< = (A_<, \overline{R}, \overline{S})$ that is stepwise $\overline{R}, \overline{S}$ homogeneous.*

We say that A has Property Ω *if $A \models \varphi$ iff A has a stepwise $\overline{R}, \overline{S}$ homogeneous submodel D such that $(D_<, \overline{R}, \overline{S}) \models \forall \overline{x} \theta$.*

Claim 1 *1. Suppose that $B_< = (A_<, \overline{R}, \overline{S})$ is stepwise $\overline{R}, \overline{S}$ homogeneous, and that $(A_<, \overline{R}, \overline{S}) \models \forall \overline{x} \theta$. Then φ is valid.*

2. For all A, if A has Property Ω, then $A \models \varphi$ iff φ is valid.

To prove 1., let M be an arbitrary σ_0 structure, and $M_<$ be any ordered expansion. In the φ-game on $M_<$, $\exists$'s strategy is as follows. In round $2m-1$, he $\overline{R}_m$ colors $(M_<, R^M_1, S^M_1, \ldots, S^M_{m-1})$ according to the (homogeneous) $\overline{R}_m$ coloring of $B^{m-1}_<$. That is, he chooses R^M_m such that for all submodels $N_< \subseteq M_<$ of size $\leq k$, and all $C_< \subseteq B^{m-1}_<$, if $(M_< | N_<, S^M_1 | N_<, \ldots, S^M_{m-1} | N_<) \cong C_<$ then $(M_< | N_<, S^M_1 | N_<, \ldots, S^M_{m-1} | N_<, R^M_m | N_<) \cong (C_<, R^B_m | C_<)$. Since each such $B^{m-1}_<$ is k-σ_{m-1}-rich, this strategy is well-defined. Finally, after round $2n+1$, it is clear that the structure $M'_< = (M, \overline{R}, \overline{S})$ only realizes atomic k-$(\overline{P} \cup \overline{R} \cup \overline{R})$-types that are realized in $B_<$. Therefore $M \models \forall \overline{x} \theta$, as desired.

One direction of 2. is immediate, for if $A \not\models \varphi$, then φ is not valid. Suppose that $A \models \varphi$. Then by the definition of Property Ω, there is a stepwise $\overline{R}, \overline{S}$ homogeneous submodel D of A with an ordered expansion such that $(D_<, \overline{R}, \overline{S}) \models \forall \overline{x} \theta$. By 1., this implies that φ is valid.

It now suffices to show that there is a finite A with Property Ω. Then to obtain such an A effectively, simply enumerate all finite structures, and check them in order until finding one with the property.

Claim 2 *There is a finite A^φ with Property Ω.*

We inductively define a sequence of ordered models, $A^n_<, B^n_<, \ldots, A^0_<, B^0_<$ such that A^φ will be $B^0 | \sigma_0$. (Here B^0 is the model whose ordered expansion is $B^0_<$.) Let $A^n_<$ be a k-σ_n-rich structure. For all $m \leq n$, given $A^m_<$, let $B^m_<$

be such that $B^m_< \hookrightarrow^{\overline{R}_{m+1}} A^m_<$, $sig(B^m_<) = \sigma_m$. [Note that it is an immediate consequence of the Nešetřil and Rödl theorem that $B^m_<$ can be found effectively. Since it is decidable, given two models B and A, whether $B \hookrightarrow^{\overline{R}_{m+1}} A$, one simply enumerates all the finite models, $B_1, \ldots$, and searches until a B_k such that $B_k \hookrightarrow^{\overline{R}_{m+1}} A$ is found.] Given $B^m_<$, define $A^{m-1}_<$ to be $B^m_< | \sigma_{m-1}$. Finally, let $A^\varphi = B^0 | \sigma_0$. Observe again that there is an algorithm for finding A^φ. It may be helpful to note that the construction has the following properties.

1. Each $A^m_<$ has signature σ_m and is k-σ_m-rich.
2. For $l < m \leq n$, $A^m_< | \sigma_l$ is a submodel of $A^l_<$.
3. Each $A^m_<$ is a submodel of $B^m_<$.
4. For all m, $A^{m-1}_<$ and $B^m_<$ have the same universe.

We now show that A^φ has Property Ω. If $A^\varphi \not\models \varphi$, then φ is not valid and, by Claim 1, there is no stepwise $\overline{R}, \overline{S}$ homogeneous model that satisfies $\forall \overline{x} \theta$. Suppose that $A^\varphi_< \models \varphi$, where $A^\varphi_< = B^0_< | \sigma_0$. We must demonstrate that there is a stepwise $\overline{R}, \overline{S}$ homogeneous expansion $(D_<, \overline{R}, \overline{S})$ of a submodel $D_<$ of $A^\varphi_<$, such that $(D_<, \overline{R}, \overline{S}) \models \forall \overline{x} \theta$. We define a sequence of models $C^i_<, D^i_<, C^{i'}_<, i \leq n+1$ such that the $C_<$'s are expansions of A^φ and the $D_<$'s are submodels of $A^\varphi_<$, such that for all $i < j$, $D^j_< \subset D^i_<$.

Let $C^1_< = (A^\varphi_<, \overline{R}^A_1)$ be an expansion of $A^\varphi_<$ such that $C^1_< \models \forall \overline{S}_1 \exists \overline{R}_2 \ldots \forall \overline{x} \theta$. By the construction, there is a $D^1_< \subseteq A^\varphi_<, D^1_< \cong A^0_<$, such that $\overline{R}^A_1 | D^1_<$ is a homogeneous $\overline{R}_1$-coloring of $D^1_<$. By the definition of $A^0_<$, there is an $\overline{S}_1$ expansion of $(D^1_<, \overline{R}^A_1 | D^1_<)$ isomorphic to $B^1_<$. Let $C^{1'}_< = (A^\varphi_<, \overline{R}^A_1, \overline{S}^A_1)$ be any $\overline{S}_1$ expansion of $C^1_<$ such that $(D^1_<, \overline{S}^A_1 | D^1_<)$, is isomorphic to $B^1_<$. Observe that $C^{1'}_< \models \exists \overline{R}_2 \forall \overline{S}_2 \ldots \forall \overline{x} \theta$. Let $C^2_<$ be an $\overline{R}_2$ expansion of $C^{1'}_<$ such that $C^2_< \models \forall \overline{S}_2 \ldots \forall \overline{x} \theta$. Let $D^2_<$ be a submodel of $D^1_<$ such that $(D^2_<, \overline{S}^A_1 | D^2_<) \cong A^1_<$ and $\overline{R}^A_2 | D^2_<$ is a homogeneous coloring of $(D^2_<, \overline{S}^A_1 | D^2_<)$. By iterating this procedure, we get a submodel $D^{n+1}_<$ of $A^\varphi_<$ such that $(D^{n+1}_<, \overline{R}^A | D^{n+1}_<, \overline{S}^A | D^{n+1}_<)$ is stepwise $\overline{R}, \overline{S}$ homogeneous.

This proves the claim and also the proposition. ■

Observe that the proposition immediately implies that SO($\exists$) has the finite model property.

It would be interesting to know whether this result could be strengthened in some way, e.g., by extending it to stronger logics. Certain known results establish that for some natural extensions of SO($\exists$), the satisfiablity problem becomes undecidable. Blass and Gurevich [4] proved that if one allows function symbols in the signature, then already for $\Pi^1_1(\exists)$ both the satisfiability and validity problems are undecidable. They also observe that $\Pi^1_1(\exists)$ does have the finite model property, which they attribute to Mal'tsev. Another question is whether the implication problem for SO($\exists$) is decidable. That is, given $\theta, \varphi \in$ SO($\exists$), does θ imply φ? Shmueli [26] proved a stronger negative result, that for Datalog, a sublogic of $\Pi^1_1(\exists)$, the implication problem is undecidable. This immediately implies that already the satifiability problem for the language $\mathcal{B}.\Pi^1_1(\exists)$, the set of Boolean combinations of $\Pi^1_1(\exists)$ sentences, is undecidable. On the other hand, it is easy to show that $\mathcal{B}$.SO($\exists$) still has the finite submodel property (see Theorem 3 below). Clearly together these results imply that there is no recursive function $f(x)$ such that for all $\varphi \in \mathcal{B}.\Pi^1_1(\exists)$, φ is satisfiable iff it has a model of size $\leq f(\varphi)$.

The finite submodel property

The following theorem says that SO($\exists$) has the finite submodel property.

Theorem 3 *(Mal'tsev [20]; Kreisel and Krivine [18]) Let φ be a sentence in* SO($\exists$). *For all A, if $A \models \varphi$, then there is a finite submodel $B \subseteq A$ such that $B \models \varphi$.*

They both actually prove the result stated below, which immediately yields the preceding theorem. For the sake of completeness, we include the proof of Kreisel and Krivine.

Proposition 6 *For all $\varphi \in$* SO($\forall$), *φ is equivalent to a set of $FO(\forall)$ sentences.*

Proof. We argue by induction on the number of SO quantifiers. If there are none, then the claim is obvious. Suppose, then, that $\varphi = \mathcal{Q}_n R_n \ldots \mathcal{Q}_1 R_1 \forall \overline{x} \theta$, where each $\mathcal{Q}_i$ is either $\exists$ or $\forall$, and θ is quantifier free. Let $\overline{P} \cup \{R_1, \ldots, R_n\}$ be the set of relation symbols occuring in θ. For all $m, 0 \leq m \leq n$, let $\sigma_m = \overline{P} \cup \{R_{m+1}, \ldots, R_n\}$, (so that $\sigma_n = \overline{P}$). For all m, let $\mathcal{L}^m$ be the set of FO sentences containing only relation symbols from σ_m. Let $T^0 = \{\theta^0_0, \theta^0_1, \ldots\}$ be an enumeration of the set $\{\psi \mid \psi \in \mathrm{FO}(\forall) \cap \mathcal{L}^0$ and $\forall \overline{x} \theta \models \varphi\}$. Clearly $\forall \overline{x} \theta$ is equivalent to T^0.

For the induction step, suppose that $\mathcal{Q}_{n-1} R_{n-1} \ldots \mathcal{Q}_1 R_1 \forall \overline{x} \theta$ is equivalent to an infinite set, $T^{n-1} = \{\theta^{n-1}_0, \theta^{n-1}_1, \ldots\}$ of sentences in $\mathrm{FO}(\forall) \cap \mathcal{L}^{n-1}$. Without loss of generality, we can assume that T^{n-1} is closed under logical consequence. We consider two cases.

One, $\mathcal{Q}_n$ is $\exists$. Then for all models $A, A \models \exists R_n (\bigwedge_{i \in \omega} \theta^{n-1}_i)$ iff there is a (B, R^B_n) such that $A \subseteq B$ and $(B, R^B_n) \models \bigwedge_{i \in \omega} \theta^{n-1}_i$. (Here we implicitly use the fact that $Mod(\varphi)$ is closed under substructures). By the compactness theorem, such a (B, R^B_n) exists iff for all $\theta \in T^{n-1} \cap \mathcal{L}^n, A \models \theta$. Therefore let $T^n = T^{n-1} \cap \mathcal{L}^n$.

Two, $\mathcal{Q}_n$ is $\forall$. We know that φ is equivalent to $\forall R_n (\bigwedge_{i \in \omega} \theta^{n-1}_i)$, which is equivalent to $\bigwedge_{i \in \omega} (\forall R_n \theta^{n-1}_i)$. It is easy to see that each conjunct, $\forall R_n \theta^{n-1}_i$ is equivalent to a sentence $\theta^n_i \in \mathrm{FO}(\forall) \cap \mathcal{L}^n$ such that $qr(\theta^n_i) = qr(\theta^{n-1}_i)$. [$qr$ is quantifier rank.] Therefore let $T^n = \{\theta^n_0, \theta^n_1, \ldots\}$. ∎

We now observe some easy corollaries which indicate the strength of the finite submodel property. The first is a form of (logical) compactness. Here we do consider infinite signatures.

Corollary 1 *Let Γ be a set of* SO($\exists$) *sentences over a purely relational vocabulary of cardinality κ. If each $\gamma \in \Gamma$ is satisfiable, then there is a model A of cardinality $max(\kappa, \omega)$ such that $A \models \Gamma$.*

Proof. By Theorem 2 or 3, for each satisfiable $\gamma \in \Gamma$, there is a finite A_γ such that $A_\gamma \models \gamma$. Let A be the disjoint union of these structures, $A = \bigcup_{\gamma \in \Gamma} A_\gamma$. ■

The next observation provides some additional information. (Allowing constants, the question becomes trivial.)

Observation 1 *For all infinite cardinals κ, λ, if $2^\kappa < \lambda$, then there is a consistent set of* SO($\exists$) *[in fact,* FO($\exists$)*] sentences S containing no constants, of size λ, that has no model of cardinality κ.*

Proof. Suppose that κ, λ are infinite cardinals such that $2^\kappa < \lambda$. Let $\sigma = \{P_\alpha \mid \alpha < \lambda\}$ be a set of unary relation symbols, and let $S = \{\exists x(P_\alpha x \wedge \neg P_\beta x) \mid \alpha < \beta < \lambda\}$. Given any model A such that $|A| = \kappa$, let $A_\alpha = \{a \in A \mid A \models P_\alpha a\}$. Because there are only 2^κ subsets of A, there are α, β such that $A_\alpha = A_\beta$. Therefore $A \not\models \exists x(P_\alpha x \wedge \neg P_\beta x)$. ■

The following result improving Corollary 1, and the converse of the preceding Observation, is due to Joel David Hamkins.

Theorem 4 *(Hamkins) For all infinite cardinals κ, λ, if $2^\kappa \geq \lambda$, then every set of satisfiable* SO($\exists$) *sentences, without constants, of cardinality λ has a model of cardinality κ.*

Proof. First observe that, without loss of generality, we can assume that $\lambda = 2^\kappa$, that σ is a vocabulary consisting of λ many n-ary relations for each n, and that S is the set of all consistent SO($\exists$) sentences over σ. Second, by the finite model property for SO($\exists$), each $\varphi \in S$ is implied by some consistent FO($\exists$) sentence, so it suffices to show that there is a model of cardinality κ satisfying each such FO($\exists$) sentence.

Observe also that it suffices to show that for each n, the set of consistent sentences $\{\varphi \mid \varphi = \exists x_0 \ldots x_{n-1}(\bigwedge_{i<j} x_i \neq x_j \wedge \psi(\overline{x}))\}$, where $\psi(\overline{x})$ is a conjunction of atomic and negative atomic formulas, has a model A_n of size κ. For then the model $A = \bigcup_n A_n$, the disjoint union of all the A_n, has the desired property. Furthermore, we claim that it suffices to prove the result for $n = 1$. To generalize to each larger n, we treat the universe as a set of κ many pairwise disjoint n-tuples, and view each atomic formula with free variables from among $x_0, \ldots, x_{n-1}$ as a unary predicate on each such n-tuple.

We now prove the case for $n = 1$. To simplify the notation, we identfy λ with ${}^\kappa 2$, the set of all functions from κ into $\{0, 1\}$. Let $\alpha_f(x), f \in \lambda$, be the set of all atomic formulas with one free variable, that is, $\alpha_f(x) = P(x, \ldots, x)$, where P is any relation in σ. Each element of A_1 will be a function in ${}^{\lambda}2$, and the interpretation of each atomic formula α_f is determined as follows. For each α_f and each $a \in A_1$, $A \models \alpha_f(a)$ iff $a(f) = 1$. (Recall that a is actually a function on λ.) All that remains is to show that there is a set $A_1 \subset {}^{\lambda}2$, of size κ, such that for each finite set of functions $f_1, \ldots f_l, g_1, \ldots, g_m$ in λ, there is an $a \in A_1$ such that for all $i \leq l, a(f_i) = 1$ and all $j \leq m, a(g_j) = 0$. [Note that this is equivalent to the topological fact that ${}^{\lambda}2$ has a dense subset of size κ. This is the key idea in the proof.]

Let $A_1 \subset {}^{\lambda}2$ be the following set.

$$\{a \in {}^{\lambda}2 \mid \exists \text{ finite set } U \in \kappa \text{ such that } \forall f, g \in \lambda \text{ if } f|U = g|U \text{ then } a(f) = a(g)\}$$

It is clear that $|A| = \kappa$. Let $f_1, \ldots f_l, g_1, \ldots, g_m$ be as above, and let $U \subset \kappa$ be a finite set such that for each pair f_i, g_j there is an element $u \in U$ such that $f_i(u) \neq g_j(u)$. Then it is easy to see that there is an $a \in A_1$ that is 'closed under functions restricted to U' such that for all $i, a(f_i) = 1$ and for all $j, a(g_j) = 0$. ■

We write $A \equiv_{SO(\exists)} B$ $[(A, \overline{a}) \equiv_{SO(\exists)} (B, \overline{b})]$ iff for all $\varphi \in$ SO($\exists$) $[\theta(\overline{x}) \in$ SO($\exists$)], $A \models \varphi$ iff $B \models \varphi$ $[A \models \theta(\overline{a})$ iff $B \models \theta(\overline{b})]$. Also, for $B \subseteq A$, let $B \preceq_{SO(\exists)} A$ iff for all tuples $\overline{b} \subseteq B$ and all formulas $\theta(\overline{x}) \in$ SO($\exists$), $A \models \theta(\overline{b})$ iff $B \models \theta(\overline{b})$. Likewise, define $A \equiv_{FO(\exists)} B$ and $B \preceq_{FO(\exists)} A$. By the finite submodel property, it is clear that for all A and B, $A \equiv_{SO(\exists)} B$ iff $A \equiv_{FO(\exists)} B$ $[A \preceq_{SO(\exists)} B$ iff $A \preceq_{FO(\exists)} B]$. In particular, every complete SO($\exists$) $\cup$ SO($\forall$) theory is equivalent to a set of FO($\exists$)$\cup$FO($\forall$) sentences. This also yields the following, by using the analagous result for first order logic.

Corollary 2 [Downward Löwenheim-Skolem property] *Let A be an infinite model, with signature σ of size κ. Then there is a $B \subseteq A, |B| \leq \kappa$ such that $B \preceq_{SO(\exists)} A$.*

We now turn briefly to discuss SO($\forall$) and compactness. By Proposition 6 and the compactness theorem for first order logic, the logic SO($\forall$) satisfies the following form of the compactness theorem. Any set Γ of SO($\forall$) sentences is satisfiable iff only every finite subset of Γ is satisfiable. Another statement of (logical) compactness asserts that if for all pairs of sets of sentences Γ, Δ, if $\Gamma \models \Delta$ then there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \models \Delta'$. (Recall that $\Gamma \models \Delta$ iff for all models A, if A satisfies every $\gamma \in \Gamma$, then it satisfies some $\delta \in \Delta$.) These two notions are clearly equivalent for logics that are closed under negation, but not for SO($\exists$) and SO($\forall$). This follows immediately from Compton's result [6] that already Datlog($\neg$) fails to have this form of compactness. For example, over $\sigma = \{s, t, Exy\}$, let Δ consist of the single sentence $\delta \in$FO($\exists$), $\delta = \exists xyz(Etx \vee (y \neq z \wedge Exy \wedge Exz))$, and let $\Gamma = \{\gamma_0, \gamma_1, \ldots\}$, where $\gamma_0 \in \Pi^1_1(\exists)$ says that there is a path from s to t and for all $i \geq 1$, $\gamma_i \in$FO($\exists$) says that there is a path of length i rooted at s. Then clearly $\Gamma \models \Delta$, though this does not hold for any finite $\Gamma' \subseteq \Gamma$.

4 Definability and non-definability

4.1 Non-definability via Ramsey theory

Following Fagin's theorem that $\Sigma^1_1 = \mathrm{NP}$, there has been a great deal of work in finite model theory that shows that various properties that are known to be in NP (that is, in Σ^1_1) are not definable in certain fragments of Σ^1_1 (e.g. see [2], [3], and [9]). These fragments are defined in terms of a restriction on the use of SO quantification, and the results are generally proved using either a modified Ehrenfeucht-Fraïssé game, or some version of Hanf's lemma. In this section, we continue our investigation of the languages $\Pi^1_1(\exists)$ and $\mathrm{SO}(\exists)$, which limit the use of FO quantification. The main result, which uses the theorem of Nešetřil and Rödl, is that there is a property expressible in Σ^1_1 that is closed under extensions but not definable in $\mathrm{SO}(\exists)$. This result can also be viewed as saying that a certain existential preservation theorem fails for Σ^1_1 and for SO.

We first define the Σ^1_1 property that is closed under extensions that we will prove is not expressible in $\mathrm{SO}(\exists)$. Let $\sigma = \{Exy, Rxy\}$. A *(directed) cycle′* in a model is a sequence of distinct elements $(a_0, \ldots, a_n), n \geq 2$, such that for all $m < n, Ea_m a_{m+1}$; $Ea_n a_0$; and for all $i \neq j, Ra_i a_j$. It is clear that the class of models that contain a cycle′ is closed under extensions and definable in Σ^1_1. Notice also that every cycle′ is irreducible; this is why we need the relation Rxy. This will allow us to apply the stronger form of the Nešetřil and Rödl theorem over classes of the form $Soc(\mathcal{A})$, where $\mathcal{A}$ will be the class of all (expansions of) cycle′s in $\mathcal{O}_\tau$.

Theorem 5 *There is a class $\mathcal{C} \in \Sigma^1_1$ that is closed under extensions that is not definable by any* $\mathrm{SO}(\exists)$ *sentence.*

Proof. As in the proof of Theorem 2, it is easier to work with $\mathrm{SO}(\forall)$, so we will show that the class of cycle′-free models is not in $\mathrm{SO}(\forall)$. In fact, in order to be able to apply the Nešetřil and Rödl theorem more directly, we will prove this claim over the class $\mathcal{O}_\tau$, $\tau = \sigma \cup \{<\}$, which obviously implies the general result. More precisely, letting $\mathcal{C} = \{A \mid A \in \mathcal{O}_\tau$ and A contains a cycle′$\}$ and $\overline{\mathcal{C}} = \mathcal{O}_\tau \setminus \mathcal{C}$, we prove that there is no $\varphi \in \mathrm{SO}(\forall)$ such that $Mod(\varphi) \cap \mathcal{O}_\tau = \overline{\mathcal{C}}$.

We require the following definition, a refinement of the notion of k-$\overline{P}$-rich.

Definition 6 *Let τ be a signature containing $<$, ρ a signature disjoint from τ, and $k \in \omega$. For all $A \in \mathcal{O}_\tau$ and all $B \in \mathcal{O}_{\tau\cup\rho}$, we say that B is k-ρ-A-rich iff*

1. *for all $C \in \mathcal{O}_{\tau\cup\rho}, |C| = k$, if $C|\tau$ is isomorphic to a submodel of A, then C is isomorphic to a submodel of B;*

2. *for all $B' \subseteq B$ such that $B'|\tau$ is irreducible, there is an $A' \subseteq A$ such that $A' \cong B'|\tau$.*

It is easy to see that for any ordered A, there are finite k-ρ-A-rich structures.

The basic idea of the proof is as follows. We argue by contradiction. Suppose that $\varphi \in \mathrm{SO}(\forall)$ defines $\overline{\mathcal{C}}$, and let k be the number of FO quantifiers in φ. First we define a model $A^\varphi \in \overline{\mathcal{C}}$, using a construction similar to that in the proof of Theorem 2, such that the existence of a winning strategy for $\exists$ in the φ-game on A^φ implies that $\exists$ has a 'homogeneous' winning strategy on a sufficiently 'k-rich' expansion of a submodel A_1^φ of A^φ. As before, there will be something like a 'stepwise $\overline{R}, \overline{S}$ homogeneous expansion' of A_1^φ that satsifies the FO part, $\forall\overline{x}\theta$, of φ. We then choose $B \in \mathcal{C}$ to be a simple cycle$'$ of length $k+1$, containing no shorter cycle$'$s. By the 'richness' of A_1^φ, every submodel $B' \subseteq B$ of size k is isomorphic to a submodel of A_1^φ. Using his homogeneous winning strategy on A_1^φ, $\exists$ can also win the φ-game on B. Thus $B \models \varphi$, contradicting the assumption that φ defines $\overline{\mathcal{C}}$.

To simplify the proof, we assume that $\varphi \in \Pi_2^1(\forall), \varphi = \forall\overline{S}\exists\overline{R}\forall\overline{x}\theta$, and $k = length(\overline{x})$. [To generalize the argument, one just iterates the construction of A^φ as in the proof of Theorem 2.] Let $\mathcal{E} = \{B \mid B \in \overline{\mathcal{C}} \text{ and } |B| = k\}$; and let A_0^φ be a model in $\overline{\mathcal{C}}$ such that every $B \in \mathcal{E}$ is isomorphic to a submodel of A_0^φ. For example, one can take the disjoint union of all $B \in \mathcal{C}$, up to isomorphism, and then extend $\{<\}$ to be a total order. Observe that here every irreducible submodel of A_0^φ is a submodel of some $B \in \mathcal{E}$, so that A_0^φ is indeed cycle$'$-free.

Let $(A_1^\varphi, \overline{S}^{A_1}), A_1^\varphi \in \overline{\mathcal{C}}$, be a k-$\overline{S}$-A_0-rich model. Let $\mathcal{A} = \{A \mid A \text{ is a } \tau \cup \overline{S}$ model and $A|\tau \in \mathcal{O}_\tau$ is a cycle$'\}$. Observe that every $A \in \mathcal{A}$ is an irreducible model, so we can apply the Nešetřil and Rödl theorem to $Soc(\mathcal{A})$ (implicitly invoking the correspondence between models and set systems). Observe that $(A_1^\varphi, \overline{S}) \in Soc(\mathcal{A})$. In fact, $Soc(\mathcal{A})$ is just the class of all $\overline{S}$ expansions of cycle$'$ free models in $\overline{\mathcal{C}}$. Therefore, by the Nešetřil and Rödl theorem, there is an $(A^\varphi, \overline{S}^A) \in Soc(\mathcal{A})$ such that $(A^\varphi, \overline{S}^A) \hookrightarrow^{\overline{R}} (A_1^\varphi, \overline{S})$. Note that $A^\varphi \in \overline{\mathcal{C}}$. This completes the construction.

By hypothesis, since $A^\varphi \in \overline{\mathcal{C}}, A^\varphi \models \varphi$ and $(A^\varphi, \overline{S}^A) \models \exists\overline{R}\forall\overline{x}\theta$. Let $(A^\varphi, \overline{S}^A, \overline{R}^A)$ be an $\overline{R}$ expansion such that $(A^\varphi, \overline{S}^A, \overline{R}^A) \models \forall\overline{x}\theta$. By the construction of A^φ, there is a submodel $A_1 \subseteq A^\varphi$, $(A_1, \overline{S}^A|A_1) \cong (A_1^\varphi, \overline{S}^{A_1})$, such that $\overline{R}^A|A_1$ homogeneously colors $(A_1, \overline{S}^A|A_1)$. Since $\forall\overline{x}\theta$ is preserved under substructures, also $(A_1, \overline{S}^A|A_1, \overline{R}^A|A_1) \models \forall\overline{x}\theta$.

Finally, let $B \in \mathcal{C}$ be a cycle$'$ of length $k+1$. We claim that $B \models \varphi$, that is, $B \models \forall\overline{S}\exists\overline{R}\forall\overline{x}\theta$. Let $(B, \overline{S}^B)$ be any expansion of B. Observe that every $B' \subseteq B, |B'| = k$, is in $\mathcal{E}$, and hence isomorphic to a submodel of A_0^φ. As $(A_1^\varphi, \overline{S}^A)$ is k-A_0^φ-$\overline{S}$-rich, $(B', \overline{S}^B|B')$ is also isomorphic to a submodel of $(A_1, \overline{S}^A|A_1)$. Then, since $\overline{R}^A|A_1$ is a homogeneous coloring of $(A_1, \overline{S}^A|A_1)$, it determines a unique $\overline{R}$ expansion $(B, \overline{S}^B, \overline{R}^B)$ such that for every $B' \subseteq B$ with $|B'| = k$, $(B', \overline{S}^B|B', \overline{R}^B|B')$ is isomorphic to a submodel of $(A_1, \overline{S}^A|A_1, \overline{R}^A|A_1)$. Therefore $(B, \overline{S}^B, \overline{R}^B) \models \forall\overline{x}\theta$, and $(B, \overline{S}^B) \models \exists\overline{R}\forall\overline{x}\theta$. As $\overline{S}^B$ was an arbitrary

$\overline{S}$ expansion of B, this proves that $B \models \forall\overline{S}\exists\overline{R}\forall\overline{x}\theta$, contradicting the assumption that φ defines $\overline{\mathcal{C}}$. ■

We now observe a few corollaries of the above proof. Recall that a homomorphism from A to B is a function $f : A \longrightarrow B$ such that for all k-ary relations $R \in sig(A)$ and all k-tuples $\overline{a}$ in A, if $A \models R\overline{a}$, then $B \models R(f(\overline{a}))$. (Here, f is not required to be either injective or surjective.) Obviously any class closed under homomorphisms is also closed under extensions. Again, let $\sigma = \{Exy, Rxy\}$ and let $\mathcal{C}' = \{A \mid A$ contains a cycle$'$ or $A \models \exists x Exx\}$. It is easy to see that $\mathcal{C}'$ is closed under homomorphisms and definable in Σ^1_1. By the preceding proof, $\mathcal{C}'$ is not definable in SO($\exists$). Therefore we have the following strengthening of the previous theorem.

Corollary 3 *There is a class $\mathcal{C} \in \Sigma^1_1$ that is closed under homomorphisms that is not definable by any* SO($\exists$) *sentence.*

Examining the proof of Theorem 5, it is clear that we only used the following properties of the class $\mathcal{C}$ of structures containing cycle$'$s. One, it is closed under extensions and contains arbitrarily large 'minimal models', i.e., models A such that for all proper submodels $B \subseteq A$, $B \notin \mathcal{C}$. Two, every $A \in \mathcal{C}$ has an irreducible submodel $B \subseteq A$ that is also in $\mathcal{C}$. (Equivalently, every minimal model of $\mathcal{C}$ is irreducible.) Therefore the proof yields the following result.

Corollary 4 *Let $\mathcal{C}$ be any class of models that is closed under extensions and contains arbitrarily large minimal submodels, such that every minimal model is irreducible. Then $\mathcal{C}$ is not defined by any* SO($\exists$) *sentence.*

The proof of Theorem 5 also suffices to establish a strict hierarchy in SO($\exists$) based on the number of FO quantifiers. (In the next section, we prove a strengthening of this result, that requires some new definitions and further argument.) Define SO($k \cdot \exists$) to be the set of SO($\exists$) sentences containing at most k FO quantifiers. [This in not FO quantifier rank.] The argument shows that if A is a cycle$'$ of size $k+1$, $\mathcal{C}_A = \{B \mid A \subseteq B\}$, and $\varphi \in$ SO($k \cdot \forall$) is true in every $B \in \overline{\mathcal{C}}_A$, then also $A \models \varphi$. Since the only property of A that is used is that it is irreducible, we have essentially shown that for any irreducible $A, |A| = k+1$, $\mathcal{C}_A$ can not be defined in SO($k \cdot \exists$). (In fact, we even have that for all consistent $\varphi \in$ SO($k \cdot \exists$), $Mod(\varphi) \not\subseteq \mathcal{C}_A$.) Clearly, though, $\mathcal{C}_A$ is definable in FO($k+1 \cdot \exists$).

Corollary 5 *For all $k \in \omega$,* SO($k \cdot \exists$) $\neq$ *SO($k+1 \cdot \exists$). Therefore the fragments* SO($k \cdot \exists$), *$k \in \omega$, form a hierarchy of strictly increasing expressive power.*

More generally, by the same argument, we can establish that every consistent SO($k \cdot \exists$) sentence has a model which contains no irreducible submodels of size $> k$. On the other hand, even over the empty signature, for all $k \geq 2$ and all n, there is a consistent $\varphi_n \in m.\Pi^1_1(k \cdot \exists)$ all of whose models are of size $\geq n$. For example, $\varphi_n = \forall R_1 x \ldots \forall R_n x \theta_n \in m.\Pi^1_1(2 \cdot \exists)$, where θ_n says that two elements realize the same atomic 1-$\overline{R}$-type, has models of exactly those cardinalities that are $> 2^n$.

4.2 Definability in m.SO($\exists$)

Proposition 7 *There is a φ in $\Pi^1_1(\exists)$ that is not equivalent to any sentence in m.SO($\exists$).*

Proof. Let $\sigma = \{Exy, s, m, t\}$. Let $\mathcal{C}$ be the class of models such that there are paths from s to m and from m to t of the same length. It is easy to see that $\mathcal{C}$ can be computed by a Datalog program, and hence is definable in $\Pi^1_1(\exists)$. In fact, $\mathcal{C}$ is defined by the following sentence, with the minimal possible SO quantifier prefix.

$$\forall Rxy(\neg Rsm \vee \exists xyzw(Rxy \wedge Exz \wedge Eyw \wedge \neg Rzw) \vee Rmt)$$

Let $\mathcal{D}$ be the class of finite models A such that E^A is a simple directed path from s to t, and let $\mathcal{D}' \subseteq \mathcal{D}$ be the subset consisting of those A such that m is the midpoint of the path. Suppose, for contradiction, that $\mathcal{C}$ is defined by a m.SO($\exists$) sentence θ. Then, over the class $\mathcal{D}$, θ defines $\mathcal{D}'$. Now, view each $A \in \mathcal{D}$ as a word, in the sense of formal language theory, over the alphabet $\{m, o\}$, (which contains exactly one occurence of the letter m). By Buchi's theorem, (see [12]) every m.SO sentence defines a regular language over 'word models', as above. But $\mathcal{D}'$ is not a regular language, so it cannot be defined by any m.SO sentence.

[One can also prove the result using a simple m.SO pebble game instead of Buchi's theorem.] ∎

The next proposition separates the first two levels of the m.SO($\exists$) hierarchy.

Proposition 8 *There is a class $\mathcal{C}$ that is definable in m.$\Sigma^1_2(\exists)-$m.$\Pi^1_1(\exists)$.*

Proof. Let $\sigma = \{Exy, c_1, c_2, d_1, d_2\}$, and let $\mathcal{C}$ be the class of models over σ such that there are disjoint paths from c_1 to c_2 and from d_1 to d_2. It is known that this class is NP-complete. We first show that it is definable in m.$\Sigma^1_2(\exists)$. Let φ be the following sentence,

$$\exists S \exists T \forall P \forall Q(c_1 \neq d_1 \wedge Sc_1 \wedge Td_1 \wedge$$

$$(\neg Pc_1 \vee \neg Qd_1 \vee \exists x_1x_2(Px_1 \wedge Ex_1x_2 \wedge Sx_2 \wedge \neg Tx_2 \wedge \neg Px_2) \vee$$

$$\exists x_1x_2(Qx_1 \wedge Ex_1x_2 \wedge Tx_2 \wedge \neg Sx_2 \wedge \neg Qx_2) \vee (Ps_2 \wedge Qt_2)))$$

It is easy to check that $A \models \varphi$ iff there is an expansion (A, S, T) such that there are paths P from c_1 to c_2 [Q from d_1 to d_2], all of whose elements are in $S \wedge \neg T$ [$T \wedge \neg S$].

We prove that $\mathcal{C}$ is not in m.$\Pi^1_1(\exists)$ by showing that $\overline{\mathcal{C}}$ is not definable in m.$\Sigma^1_1(\forall)$. Suppose that $\psi = \exists \overline{R} \forall \overline{x} \theta$, with $n = length(\overline{R})$ and $k = length(\overline{x})$. It suffices to show that there is an $A \in \overline{\mathcal{C}}$ and is a $B \in \mathcal{C}$ such that if $A \models \psi$, then $B \models \psi$. Let the universe of A be $\{c_1, c_2, d_1, d_2, 0, 1, \ldots, 2^n \cdot 3k + 2\}$, and the universe of B be $A \cup \{e\}$. Let E^A be

$$\{(c_1, 0), (2^n \cdot 3k + 2, c_2)\} \cup \{(i, j) \mid i, j \in \mathbf{N} \text{ and } j = i + 1\}$$

$$\cup\{(d_1, i) \mid 1 \leq i \leq 2^n \cdot 3k + 1\} \cup \{(i, d_2) \mid 1 \leq i \leq 2^n \cdot 3k + 1\},$$

and $E^B = E^A \cup \{(d_1, e), (e, d_2)\}$. In A, c_1 and c_2 are connected by a single path of length $2^n \cdot 3k + 3$, and d_1 and d_2 are connected by $2^n \cdot 3k + 1$ paths of length 2 that each intersect the only path from c_1 to c_2. B contains a disjoint path from d_1 to d_2, through e. Hence $A \in \overline{\mathcal{C}}$ and $B \in \mathcal{C}$.

Suppose that $A \models \psi$ and $A' = (A, \overline{R})$ is an expansion such that $A' \models \forall \overline{x} \theta$. It suffices to show that there is an expansion $B' = (B, \overline{R})$ such that every k-$(\sigma \cup \overline{R})$-type that is realized in B' is also realized in A'. There are 2^n 1-$\overline{R}$-types, so there is one such type, τ, that is realized at least $3k + 1$ times among the elements $\{1, \ldots, 2^n \cdot 3k + 1\}$ of A. Let $B' = (B, R)$ be the expansion of B such that it contains $(A, \overline{R})$ as a submodel, and $tp(e) = \tau$.

Let $\overline{b} = (b_1, \ldots, b_k)$ be a k-tuple in B. If each b_i is in A, then $\overline{b}$ realizes the same type in A' and B', since the types are atomic and $A' \subseteq B'$. So suppose that some b_i is the element e and assume, without loss of generality, that $i = 1$, $\overline{b}$ is injective and $\overline{b}$ does not contain any element named by a constant. By our construction, there is a b' in $\{1, \ldots, 2^n \cdot 3k + 1\}$ of type τ that is not equal to or adjacent to any of the $b_i \in \overline{b}$. It is easy to see that $\overline{b}' = (b', b_2, \ldots, b_k)$ and $\overline{b}$ realize the same type. As observed above, the type of $\overline{b}$ is also realized [by $\overline{b}$] in A', as desired. ■

Observe that in the preceding proof, we could have used the simpler property 'there is an element a and a path from s to t that does not contain a'.

5 Finite variable $SO(\exists)$

In this section we examine a hierarchy in SO($\exists$) based on the number of FO variables that occur in a formula. Finite variable logic L^k has been studied extensively in the context of finite model theory. In some earlier papers ([24], [23]) we investigated the existential fragment $L^k(\exists)$. Here we consider SO($\exists^k$), second order $L^k(\exists)$, and prove that the fragments SO($\exists^k$), $k \in \omega$, form a strict hierarchy of increasing expressive power. We also show that if A is a σ model, and σ includes a relation symbol of arity ≥ 4, then the property of containing a submodel isomorphic to A is not definable in SO($\exists^{|A|-1}$).

We recall the following definitions. Let L^k be the set those FO formulas all of whose variables, both free and bound, are among $x_1, \ldots, x_k$. We let $L^k(\exists)$ be the set of existential formulas of L^k, that is, the closure of the set of basic L^k formulas under $\wedge, \vee$, and $\exists$. Observe that $L^k(\exists) \subseteq$FO($\exists$), though it is easy to see that for all $k \geq 2$ and all n, $L^k(\exists) \not\subseteq$FO($n \cdot \exists$). (For example, below it is shown how to say that there is a path of arbitrarily long length in $L^2(\exists)$.) Let

$L^{k,n}(\exists)$ be the set of $L^k(\exists)$ formulas with quantifier rank $\leq n$. For all A and B, $A \preceq^{k,n} B$ iff for all $\varphi \in L^{k,n}(\exists)$, if $A \models \varphi$, then $B \models \varphi$.

This relation may be characterized by the following variant of the Ehrenfeucht-Fraïssé game. The n-round $\exists^k$-game from A to B is played between a Spoiler and a Duplicator, with k pairs of pebbles $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$. In each round, the Spoiler first plays a pebble α_i (which might already have been played) on an element in A, and the Duplicator responds by placing the corresponding pebble, β_i, on an element in B. We say that the Duplicator has a winning strategy in the game if she can play so that after each round $m, m \leq n$, the current position of the pebbles on A and B determines a partial isomorphism. The following proposition expresses the connection between this game and logical definability.

Proposition 9 (Kolaitis and Vardi [17]) *For all structures A and B, the following conditions are equivalent.*

1. $A \preceq^{k,n} B$
2. *The Duplicator has a winning strategy in the n-round $\exists^k$-game from A to B.*

Definition 7 *For $k \in \omega$, let $\mathrm{SO}(\exists^k)$ be the closure of $L^k(\exists)$ under SO quantification. That is, $L^k(\exists) \subseteq \mathrm{SO}(\exists^k)$ and for all $\varphi \in \mathrm{SO}(\exists^k)$, $\exists R\varphi \in \mathrm{SO}(\exists^k)$ and $\forall R\varphi \in \mathrm{SO}(\exists^k)$.*

As in the FO case, observe that $\mathrm{SO}(\exists^k) \subseteq \mathrm{SO}(\exists)$ and $\mathrm{SO}(k \cdot \exists) \subseteq \mathrm{SO}(\exists^k)$. On the other hand, we do not know whether the latter inclusion is proper. The strictness of the $\mathrm{SO}(\exists^k)$ hierarchy will follow from our proof that every consistent $\mathrm{SO}(\exists^k)$ sentence has a model that contains no irreducible submodel of size $k+1$.

In [23], a model A is defined to be k-universal if for all consistent sentences $\theta \in L^k_{\infty\omega}(\exists)$, infinitary $L^k(\exists)$, $A \models \theta$. Equivalently, A is k-universal just in case for all B, $B \preceq^k_{\infty\omega} A$. (The existence of finite k-universal models was observed in [24]. Note that this implies that for all k and all signatures σ, there is an $n \in \omega$ such that every consistent $L^k(\exists)$ sentence over σ has a model of cardinality n.) Here we need a notion of k-universal over ordered structures, stratified in terms of quantifier rank.

Definition 8 *Let σ be a signature containing $<$. For $A \in \mathcal{O}_\sigma$, we say that A is $k^<_n$-universal iff for all $B \in \mathcal{O}_\sigma$, $B \preceq^{k,n} A$.*

It is easy to show that for all k, n, and σ, there are finite $k^<_n$-universal models that contain no irreducible submodels of size $k+1$. [For example, one can modify the construction of the k_n-universal models B^n_k from the proof of Proposition 13 in [24] to allow for the built in order.] In contrast to $L^k(\exists)$, we observe below that for all n, there is a sentence $\varphi_n \in \Pi^1_1(\exists^2)$, containing a single (binary) second order variable, that only has models of size $\geq n$. In particular, this implies that even for a very restricted set of sentences in $SO(\exists^2)$, there is no finite universal model that satisfies every sentence in the set. [We do not know whether there are finite universal models for fragments of $m.\mathrm{SO}(\exists^k)$ containing a fixed number of SO variables.]

Observation 2 *For all n, there is a $\varphi_n \in SO(\exists^2)$, containing a single binary second order variable, that only has models of cardinality $\geq n$.*

Proof. It is well known that for all n, there is a sentence θ_n in $L^2(\exists)$, that says that there is a directed path of length 2. [For the sake of completeness, we show that for all n, there is a formula $\psi_n(x) \in L^2(\exists)$ that says that there is a path of length n starting at x. Then let $\theta_n = \exists x \psi_n(x)$. Inductively define $\psi_1(x) = \exists y Rxy$ and $\psi_{n+1}(x) = \exists y(Rxy \wedge (\exists x(x = y \wedge \psi_n(x))))$.] For $n \geq 2$, let $\varphi_n = \forall Rxy(\exists xy(x \neq y \wedge (Rxy \leftrightarrow Ryx)) \vee \theta_{n-1})$. Finally, observe that for all models (A, R^A), if $(A, R^A) \models \neg\exists xy(x \neq y \wedge (Rxy \leftrightarrow Ryx))$, then $(A, R^A) \models \theta_{n-1}$ iff R^A is a linear order of length $\geq n-1$ or R^A contains a loop or a cycle of length 3, in which case (A, R^A) contains an infinite path. ∎

Theorem 6 *For all $k, \mathrm{SO}(\exists^k) \neq SO(\exists^{k+1})$. Therefore the fragments $\mathrm{SO}(\exists^k), k \in \omega$, form a hierarchy of strictly increasing expressive power.*

The theorem follows immediately from the following proposition.

Proposition 10 *For all consistent $\varphi \in \mathrm{SO}(\exists^k)$, φ has a model containing no irreducible submodel of size $k+1$.*

Proof of proposition. As in the previous proofs, it is easier to consider $\mathrm{SO}(\forall^k)$ rather than $\mathrm{SO}(\exists^k)$. Thus, we will show that for all $\varphi \in \mathrm{SO}(\forall^k)$, if φ is true in every A that does not contain an irreducible submodel of size $k+1$, then φ is valid. To simplify the notation, we will again assume that $\varphi \in \Pi^1_2(\forall^k)$; to generalize the argument one simply iterates the construction here as in the proof of Theorem 2.

To apply the Nešetřil and Rödl theorem, we will prove the proposition over the class of models with a built in order. That is, let $\varphi = \forall \overline{S} \exists \overline{R} \theta, \theta \in L^{k,n}(\forall)$, and $sig(\varphi) = \overline{P} = \overline{P}_0 \cup \{<\}$. We show that if φ holds in every $A \in \mathcal{O}_{\overline{P}}$ with no irreducible submodel of size $k+1$, then φ is valid over $\mathcal{O}_{\overline{P}}$.

Let $A' = (A, \overline{S}^A)$ be a $k_n^<$-universal $\overline{P} \cup \overline{S}$ model containing no irreducible submodel of size $k+1$. Observe that A' is also k-$(\overline{P}_0 \cup \overline{S})$-rich. We want to show that there is an $\overline{R}$ expansion of A' such that $(A, \overline{S}^A, \overline{R}^A) \models \theta$ and $\overline{R}^A$ homogeneously colors $(A, \overline{S}^A)$. Here we argue precisely as in the proof of Theorem 5.

Let $\mathcal{A} = \{B \mid sig(B) = \overline{P} \cup \overline{S}$ and $B|\overline{P} \in \mathcal{O}_{\overline{P}}$ is an irreducible submodel of size $k+1\}$. Applying the Nešetřil and Rödl theorem to the class $Soc(\mathcal{A})$ of $\mathcal{O}_{\overline{P} \cup \overline{S}}$ models, there is an $A'_1 = (A_1, \overline{S}^{A_1}), A'_1 \in Soc(\mathcal{A})$, such that $A'_1 \hookrightarrow^{\overline{R}} A'$.

By the definition of $Soc(\mathcal{A}), A_1$ contains no irreducible submodel of size $k+1$, so by hypothesis $A_1 \models \forall\overline{S}\exists\overline{R}\theta$. Thus, $(A_1, \overline{S}^{A_1}) \models \exists\overline{R}\theta$, and there is an expansion such that $(A_1, \overline{S}^{A_1}, \overline{R}^{A_1}) \models \theta$. By the definition of A_1', there is a submodel $(B_1, \overline{S}^{A_1}|B_1) \subseteq (A_1, \overline{S}^{A_1})$, isomorphic to $A' = (A, \overline{S}^{A})$, such that $\overline{R}^{A_1}|B_1$ homogeneously $\overline{R}$ colors $(B_1, \overline{S}^{A_1}|B_1)$. Since $\theta \in L^k(\forall)$ is preserved under substructures, $B_1'' = (B_1, \overline{S}^{A_1}|B_1, \overline{R}^{A_1}|B_1) \models \theta$.

This completes the construction that we need. We now want to prove that for all $C \in \mathcal{O}_{\overline{P}}$, $C \models \forall\overline{S}\exists\overline{R}\theta$. Let $(C, \overline{S}^C)$ be an arbitrary $\overline{S}$ expansion of C. It suffices to show that there is an $\overline{R}$ expansion of $(C, \overline{S}^C)$ that satisfies θ. Since B_1'' is k-$(\overline{P}_0 \cup \overline{S})$-rich and homogeneously $\overline{R}$ colored, we can take $C'' = (C, \overline{S}^C, \overline{R}^C)$ to be the unique $\overline{R}$ expansion determined by B_1'', [as in the proof of Theorem 5].

Suppose, for the sake of contradiction, that $(C, \overline{S}^C, \overline{R}^C) \models \neg\theta$. Since $\neg\theta \in L^{k,n}(\exists)$ and $B_1'' \models \theta$, by Proposition 9 the Spoiler wins the n-round $\exists^k$-game from C'' to B''. On the other hand, $B_1''|\overline{P} \cup \overline{S}$ is a $k_n^{\leq}$-universal $\overline{P} \cup \overline{S}$ model, so we know that it is instead the Duplicator who has a winning strategy in the n-round $\exists^k$-game from $C''|\overline{P} \cup \overline{S}$ to $B''|\overline{P} \cup \overline{S}$. Now, observe that for all submodels $M \subseteq B_1''|\overline{P} \cup \overline{S}$, $N \subseteq C''|\overline{P} \cup \overline{S}$, with $|M| = |N| \leq k$, if $M \cong N$, then $(M, \overline{R}^{A_1}|M) \cong (N, \overline{R}^C|N)$. This is because the interpretation of $\overline{S}^C$ was determined by the 'homogenous' interpretation $\overline{R}^{A_1}|B_1$. Therefore every partial isomorphism from $C''|\overline{P} \cup \overline{S}$ to $B_1''|\overline{P} \cup \overline{S}$ between submodels of size $j, j \leq k$, is also a partial isomorphism between their respective $\overline{R}$ expanded structures. Thus the Duplicator's winning strategy on these structures is simultaneously a winning strategy for the game played on their $\overline{R}$ expansions, C'' and B'', thereby contradicting the supposition that $C'' \models \neg\theta$. So $C \models \varphi$, which proves that φ is indeed valid over $\mathcal{O}_{\overline{P}}$. ■

Corollary 6 *Let A be any irreducible model of size k. Then $\mathcal{C}_A = \{B \mid A \subseteq B\}$ is not definable in $SO(\exists^{k-1})$ (even with a built in order).*

This result suggests the following question: Is it true that for all finite models A, the class $\mathcal{C}_A = \{B \mid A \subseteq B\}$ is not definable in $\mathrm{SO}(\exists^{|A|-1})$? Using the previous theorem, we can show that if the signature of A contains a relation symbol of arity ≥ 4, then the answer is yes.

We first sketch the basic idea of the proof, which is quite simple. By the preceding Corollary, the property of containing a subgraph isomorphic to K_l, the complete graph on l vertices, is not definable in $\mathrm{SO}(\exists^{l-1})$. By 'symmetry', it is easy to see that the same holds if we replace K_l by its complement $\overline{K_l}$, the empty graph on l vertices. For if some sentence φ defined the latter property, then the sentence $\overline{\varphi}$, obtained by adding a negation sign in front of *every* occurence of Exy in φ, would define the former [and vice-versa]. Generalizing this idea, we have that if the 'complement' of a structure A is irreducible, then also $\mathcal{C}_A$ is not definable in $\mathrm{SO}(\exists^{|A|-1})$. It now suffices to observe that for all A, if the signature of A contains a relation symbol of arity $n, n \geq 4$, then either A or its complement is irreducible.

Definition 9 *Let A be any model with signature σ. Then the* complement *of A, denoted $\overline{A}$ is the σ model with the same universe as A, such that for all m-ary $R \in \sigma$ and all m-tuples $\overline{a}$ in A, $A \models R\overline{a}$ iff $\overline{A} \models \neg R\overline{a}$.*

The next lemma is straightforward.

Lemma 1 *For all A, $\mathcal{C}_A$ is definable in* $\mathrm{SO}(\exists^k)$ *iff $\mathcal{C}_{\overline{A}}$ is definable in* $\mathrm{SO}(\exists^k)$.

Lemma 2 *Let A be a model such that $sig(A)$ contains a n-ary relation symbol R, $n \geq 4$. Then either A or $\overline{A}$ is irreducible.*

Proof. Assume that A is not irreducible, so there are $a_1, a_2 \in A$ such that for all n-tuples $\overline{c}$ containing both a_1 and a_2, $A \models \neg R\overline{c}$. $\overline{A}$ is irreducible iff for all b_1, b_2 there is a n-tuple $\overline{c}'$, with $b_1, b_2 \in \overline{c}'$, $\overline{A} \models \neg R\overline{c}'$. For each b_1, b_2 in $\overline{A}$, choose $\overline{d}$ to be any n-tuple containing a_1, a_2, b_1, and b_2. By hypothesis, $A \models \neg R\overline{d}$, so that $\overline{A} \models R\overline{d}$. Therefore $\overline{A}$ must be irreducible, as desired. ∎

The theorem follows immediately from Corollary 6 and Lemmas 1 and 2.

Theorem 7 *For all A such that $sig(A)$ contains a n-ary relation symbol, $n \geq 4$, $\mathcal{C}_A = \{B \mid A \subseteq B\}$ is not definable in $SO(\exists^{|A|-1})$.*

It is clear that the theorem fails for models with unary signatures. The above method does not seem to be sufficient to resolve the case of signatures of arity 2 or 3.

We remark that the corresponding statement for classes of the form $\mathcal{D}_A = \{B \mid B \subseteq A\}$ is false. Let $\sigma = \{Exy\}$ and let A be a finite σ model of size $2n, n \geq 2$, such that E^A is an equivalence relation with 2 classes of size n. It is easy to show that $\mathcal{D}_A$ can be defined by a $L^{n+1}(\forall)$ sentence with quantifier rank $n+1$. This idea generalizes easily to any (non-empty) signature.

6 Final remarks

We have begun investigating the existential logic $\mathrm{SO}(\exists)$. One direction for future research would be to consider stronger fragments of SO that still only define classes closed under extensions. Allowing function symbols is one possibility. Another is to introduce SO quantification over certain restricted classes of relations. For example, extend $\mathrm{SO}(\exists)$ by adding the quantifiers $\exists_{\mathcal{O}} Rxy$ and $\forall_{\mathcal{O}} Rxy$, to be interpreted as 'there is a linear order Rxy' and 'for all linear orders Rxy'. More generally, let $\mathcal{D}$ be any class of models closed under substructures such that for all $A \in \mathcal{D}$ and all cardinalities $\kappa > |A|$, there is an extension B in $\mathcal{D}$, $A \subset B$. Then the logic $\mathrm{SO}_{\mathcal{D}}(\exists)$, incorporating the 'generalized' SO quantifiers $\exists_{\mathcal{D}}$ and $\forall_{\mathcal{D}}$, also only defines classes that are closed under extensions.

In a somewhat different spirit, one could investigate the $\mathrm{SO}(\exists)$ theories of fixed infinite structures, for example, to determine whether they are decidable.

Open questions

We have no results concerning the SO($\exists$) hierarchy that are independent of complexity theoretic assumptions. The following seems to be the most natural problem.

> Prove $\Pi^1_1(\exists) \neq \mathrm{SO}(\exists)$.

More generally, it would be nice to show that the $\Pi^1_{2n+1}(\exists)$ hierarchy is strict.

There are many open, and difficult, questions about fragments of Σ^1_1 (see Fagin [8]). Perhaps the analogous questions for $\Pi^1_1(\exists)$ [$\Sigma^1_1(\forall)$] are more tractable. For example,

> Is there a property of graphs that is expressible in $\Pi^1_1(\exists)$ that is not defined by any $\Pi^1_1(\exists)$ sentence that contains only binary second order variables?

References

[1] F. Afrati, S. S. Cosmadakis, and M. Yannakakis. On Datalog vs. polynomial time. In *Proc. 10th ACM Symp. on Principles of Database Systems*, pages 113–126, 1991.

[2] M. Ajtai. Σ^1_1-formulae on finite structures. *Annals of Pure and Applied Logic*, 24:1–48, 1983.

[3] M. Ajtai and R. Fagin. Reachability is harder for directed than for undirected finite graphs. *The Journal of Symbolic Logic*, 55(1):113–150, 1990.

[4] A. Blass and Y Gurevich. Existential fixed-point logic. In E. Börger, editor, *Computation Theory and Logic* , volume 270 of *Lecture Notes in Computer Science*, pages 20–36. Springer-Verlag, 1987.

[5] S. Burr. On the computational complexity of Ramsey-type problems. In J. Nešetřil and V. Rödl, editors, *The Mathematics of Ramsey Theory*, pages 46–52. Springer-Verlag, 1990.

[6] K. Compton. A deductive system for existential least fixed point logic. *Journal of Logic and Computation*, 3:197–213, 1993.

[7] A. Dawar. A restricted second order logic for finite structures. In *Logic and Computational Complexity,* volume 960 of *Lecture Notes in Computer Science*, pages 393–413, 1995.

[8] R. Fagin. Finite-model theory—a personal perspective. *Theoretical Computer Science*, 116(1):3–31, 1993.

[9] R. Fagin, L. J. Stockmeyer, and M. Y. Vardi. On monadic NP vs. monadic co-NP. *Information and Computation*, 120(1):78–92, 1995.

[10] R. Graham, B. Rothschild, and J. Spencer. *Ramsey Theory*. John Wiley & Sons, 1980.

[11] M. Grohe. Existential least fixed-point logic and its relatives. *Journal of Logic and Computation*, 7:205–228, 1997.

[12] Y. Gurevich. Monadic second-order theories. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, pages 479–506. Springer-Verlag, 1985.

[13] D. S. Johnson. A catalog of complexity classes. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume A, Algorithms and complexity*, pages 67–161. The MIT Press/Elsevier, 1990.

[14] Ph. G. Kolaitis and J. Väänänen. Generalized quantifiers and pebble games on finite structures. *Annals of Pure and Applied Logic*, 74(1):23–75, 1995.

[15] Ph. G. Kolaitis and M. Y. Vardi. The decision problem for the probabilities of higher-order properties. In *Proc. 19th ACM Symp. on Theory of Computing*, pages 425–435, 1987.

[16] Ph. G. Kolaitis and M. Y. Vardi. 0-1 laws and decision problems for fragments of second-order logic. *Information and Computation*, 87:302–338, 1990.

[17] Ph. G. Kolaitis and M. Y. Vardi. On the expressive power of Datalog: tools and a case study. In *Proc. 9th ACM Symp. on Principles of Database Systems*, pages 61–71, 1990. Full version appeared in IBM Research Report RJ8010, March 1991.

[18] G. Kreisel and J.-L. Krivine. *Elements of Mathematical Logic*. North Holland, 1972.

[19] T. Lacoste. *Lois de convergence et lois 0-1 dans les structures aléatoires finies: une approche logique et finitiste*. PhD thesis, L.I.M., Université Aix-Marseille I, 1996.

[20] A. I. Mal′tsev. Model correspondences. In *The metamathematics of algebraic systems: Collected papers: 1936-1967*, pages 66–94. North-Holland, 1971. Translated from the Russian.

[21] J. Nešetřil and V. Rödl. Partitions of finite relational and set systems. *Journal of Combinatorial Theory A*, 22:289–312, 1977.

[22] J. Nešetřil and V. Rödl. The partite construction and Ramsey set systems. *Discrete Mathematics*, 75:327–334, 1989.