1. Cross product of two chooses:

$$T\left(\binom{\omega}{a_1} \times \binom{\omega}{a_2}\right) = \binom{a_1 + a_2}{a_1} = \frac{(a_1 + a_2)!}{a_1! a_2!}$$

2. Cross product of k chooses:

$$T\left(\binom{\omega}{a_1} \times \binom{\omega}{a_2} \times \ldots \times \binom{\omega}{a_k}\right) = \binom{a_1 + a_2 + \ldots + a_k}{a_1, a_2, \ldots, a_k} = \frac{(a_1 + a_2 + \ldots + a_k!)}{a_1! a_2! \ldots a_k!}$$

3. Yes: the problem $T(\binom{\omega^2}{a})$ can be broken down into several "smaller" problems in the format above.

4. $T(\binom{\omega^2}{a})$ is the Schroeder's fourth problem (OEIS A000311) number of $a + 1$. It's also referred to as the number of total partitions of $a + 1$. For example, $T(\binom{\omega^2}{3}) = 26$.

5. The sequence $T(\binom{\omega^3}{a})$ is solved for exactly but doesn't seem to be in the OEIS. Starting at $a = 1$, the sequence is 1, 14, 509, 35839, 4154652, 718142257, 173201493539, 55580900954954, 22900450653281951, 11782966082685899537 (the last term is $a = 10$). We haven't yet tried to solve for a prettier recurrence than the super-general one below; given that $T(\binom{\omega^2}{a})$ has a nicer recurrence it's reasonable to expect that $T(\binom{\omega^3}{a})$ has one as well.

6. The general recurrence for these types of problems is

$$T\left(\binom{\omega^{d_1}}{a_1} \times \binom{\omega^{d_2}}{a_2} \times \ldots \times \binom{\omega^{d_m}}{a_m}\right) \leq$$
$$\sum_{\substack{k=1 \\ d_k=1 \\ a_k \geq 0}}^{m} T\left(\binom{\omega^{d_1}}{a_1} \times \binom{\omega^{d_2}}{a_2} \times \ldots \times \binom{\omega^{d_{k-1}}}{a_{k-1}} \times \binom{\omega^{d_k}}{a_k - 1} \times \binom{\omega^{d_{k+1}}}{a_{k+1}} \times \ldots \times \binom{\omega^{d_m}}{a_m}\right) +$$
$$\sum_{\substack{k=1 \\ d_k>1}}^{m} \sum_{i=1}^{a_k} T\left(\binom{\omega^{d_1}}{a_1} \times \binom{\omega^{d_2}}{a_2} \times \ldots \times \binom{\omega^{d_{k-1}}}{a_{k-1}} \times \binom{\omega^{d_k - 1}}{i} \times \binom{\omega^{d_k}}{a_k - i} \times \binom{\omega^{d_{k+1}}}{a_{k+1}} \times \ldots \times \binom{\omega^{d_m}}{a_m}\right)$$

for all integers $d_k, a_k \geq 0$ where $\exists k\colon d_k, a_k > 0$. The base case is where $\forall k\colon d_k, a_k = 0$ where the value is 1.