Big Ramsey degrees using parameter spaces

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BIG RAMSEY DEGREES USING PARAMETER SPACES

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Abstract. We show that the universal homogeneous partial order has finite big Ramsey degrees and discuss several corollaries. Our proof uses parameter spaces and the Carlson–Simpson theorem rather than (a strengthening of) the Halpern–Läuchli theorem and the Milliken tree theorem, which are the primary tools used to give bounds on big Ramsey degrees elsewhere (originating from work of Laver and Milliken). This new technique has many additional applications. To demonstrate this, we show that the homogeneous universal triangle-free graph has finite big Ramsey degrees, thus giving a short proof of a recent result of Dobrinen.

1. Introduction

We consider graphs, partial orders, (vertex)-ordered graphs and partial orders with linear extensions as special cases of model-theoretic relational structures (defined in Section 2). Given structures $\mathbf{A}$ and $\mathbf{B}$, we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all embeddings from $\mathbf{A}$ to $\mathbf{B}$. We write $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,l}$ to denote the following statement:

> For every colouring χ of $\binom{\mathbf{C}}{\mathbf{A}}$ with r colours, there exists an embedding $f\colon \mathbf{B} \to \mathbf{C}$ such that χ does not attain more than l values on $\binom{f(\mathbf{B})}{\mathbf{A}}$.

For a countably infinite structure $\mathbf{B}$ and its finite substructure $\mathbf{A}$, the *big Ramsey degree* of $\mathbf{A}$ in $\mathbf{B}$ is the least number $L \in \omega \cup \{\omega\}$ such that $\mathbf{B} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,L}$ for every $r \in \omega$; see [KPT05]. A countably infinite structure $\mathbf{B}$ has *finite big Ramsey degrees* if the big Ramsey degree of $\mathbf{A}$ in $\mathbf{B}$ is finite for every finite substructure $\mathbf{A}$ of $\mathbf{B}$.

A countable structure $\mathbf{A}$ is called *(ultra)homogeneous* if every isomorphism between finite substructures extends to an automorphism of $\mathbf{A}$. It is well known that there is an (up to isomorphism) unique homogeneous partial order $\mathbf{P}$ with the property that every countable partial order has an embedding to $\mathbf{P}$. We call $\mathbf{P}$ the *universal homogeneous partial order*. Similarly, there is an up to isomorphism unique homogeneous triangle-free graph $\mathbf{H}$ (called the *universal homogeneous triangle-free graph*, sometimes also *triangle-free Henson graph*) such that every countable triangle-free graph embeds to $\mathbf{H}$. (See e.g. [Mac11] for more background on homogeneous structures.)

Our main result is the following.

Theorem 1.1. *The universal homogeneous partial order has finite big Ramsey degrees.*

Presently, there are just a few examples of structures with finite big Ramsey degrees known. As we show in Section 6 the universal homogeneous partial order represents an important new example of a structure in which many of the known examples can be interpreted and thus follow as a direct consequence.

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The study of big Ramsey degrees originates in work of Laver who, in 1969, showed that the big Ramsey degrees of the order of rationals are finite [Dev79, page 73], see also [EH74, Lav84]. In his argument, he re-invented the Halpern–Läuchli theorem [HL66]. His technique was later formulated more generally by means of the Milliken tree theorem [Mil79] and the notion of envelopes and embedding types [Tod10, Chapter 6]. The majority of existing results in the area continue to use the Milliken tree theorem as the primary proof technique. In particular, Devlin in 1979 [Dev79] refined Laver's argument thereby giving a precise characterisation of the big Ramsey degrees of the order of rationals. In 2006, Sauer [Sau06] and Laflamme, Sauer, and Vuksanovic [LSV06] characterised the big Ramsey degrees of the Rado graph (with precise counts given by Larson [Lar08]). This was further generalised in several followup papers [LNVTS10, DLS16].

Our proof of Theorem 1.1, for the first time in the area, uses spaces described by parameter words. This leads to a finer control over the sub-trees compared to the aforementioned constructions. Our main Ramsey tool, formulated as Theorem 2.1, is an infinitary extension of the Graham–Rothschild theorem [GR71] and is a direct consequence of the Carlson–Simpson theorem [CS84]. While the connections of the Carlson–Simpson theorem, Halpern–Läuchli theorem for trees with bounded branching and the Milliken tree theorem are well known [CS84, DK16], the additional invariants parameter spaces can preserve have been not applied in this context so far.

The proof technique presented in this paper is flexible and can be used to obtain additional finite big Ramsey degrees results for restricted structures (that is, structures omitting given substructures or satisfying certain axioms). To demonstrate this, we give a new short proof of the following recent result of Dobrinen [Dob20a]:

Theorem 1.2 (Dobrinen [Dob20a])**.** *The universal homogeneous triangle-free graph has finite big Ramsey degrees.*

Both results have well known finitary counterparts. Given a class $\mathcal{K}$ of structures, the *(small) Ramsey degree* of $\mathbf{A}$ in $\mathcal{K}$ is the least $l \in \mathbb{N} \cup \{\omega\}$ such that for every $\mathbf{B} \in \mathcal{K}$ and $r \in \mathbb{N}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,l}$. A class $\mathcal{K}$ of finite structures is *Ramsey* (or has the *Ramsey property*) if the small Ramsey degree of every $\mathbf{A} \in \mathcal{K}$ is one. The Ramsey property for finite partial orders with linear extensions was announced by Nešetřil and Rödl in 1984 [NR84] with the first proof published year later [PTW85]. The Ramsey property of finite ordered triangle free graphs is a direct consequence of the Nešetřil–Rödl theorem [NR77].

While there is a general framework which can be used to show that a given class $\mathcal{K}$ is Ramsey [HN19], the situation is very different in the context of big Ramsey degrees as there are still only a handful of structures where big Ramsey degrees are understood. The main difference is the lack of an infinite variant of the (Nešetřil and Rödl's) partite construction [NR89] (see [NR18] for its adaptation to partial orders) which has proved to be a very versatile tool in the structural Ramsey theory. Partite construction separates the structural and Ramsey arguments. Ramsey objects are constructed by sequences of (structural) products and amalgamations which are derived by a combination of Ramsey and Hales–Jewett theorems. In this respect it differs from other proofs of these results which generally represents a structural object by means of tools provided by "unstructured" Ramsey theorems (see [Prö13, Theorem 12.13] for Ramsey property of ordered graphs and [Fou97] for Ramsey property of partial orders with linear extensions).

For several decades, it was not clear how to generalize Laver's proof to (countable) restricted structures or structures in non-binary languages. Dobrinen's recent proof of Theorem 1.2 started a significant progress. Her proof uses a new method of bounding big Ramsey degrees inspired by Harrington's proof of the Halpern–Läuchli theorem, which uses techniques from forcing and the Erdős–Rado theorem. The main pigeonhole argument is a technically challenging structured tree theorem, where the tree is built using a particular enumeration of the graph Henson $\mathbf{H}$ in which certain tree levels are coding (and contain vertices of the graph being represented) while others are branching. This method was later generalized to (non-oriented) Henson graphs [Dob19]. Recently, Zucker simplified it and further generalized to finitely constrained free amalgamation classes of structures in binary languages [Zuc20]. Zucker's proof is still based on a structured pigeonhole proved by forcing techniques, but it greatly simplifies the trees by eliminating distinction between coding and branching levels. This simplification comes at a cost; the upper bounds on big Ramsey degrees obtained from Zucker's proof are bigger than ones obtained from the proof by Dobrinen (which are conjectured to be tight [Dob20a, Section 10]).

By unrelated techniques, free amalgamation classes with the property that the big Ramsey degree of a vertex is equal to one were recently characterised by Sauer [Sau20]. Bounds on big Ramsey degrees of unrestricted structures with arities greater then 2 were announced in [BCH$^+$19] with a proof based on the vector (or product) form Milliken tree theorem [BCH$^+$20b, BCH$^+$20a]. Independently, similar results were obtained by Coulson, Dobrinen and Patel [CDP20] using Dobrinen's method of strong coding trees [Dob20b].

We shall also remark that Theorem 2.1 has a direct proof based on Theorem 2 of [Kar13]. Consequently we obtain the first direct (and simpler) proof of Theorem 1.2.

The paper is organised as follows. In Section 2 we introduce parameter spaces. In Section 3 we introduce the corresponding notion of envelopes and embedding types. In Section 4 we prove the main results of this paper. In Section 5 we show that the construction is tight for determining small Ramsey degrees and thus give a new proof of a special case of the Nešetřil–Rödl theorem [NR77]. In Section 6 we discuss several corollaries. In Section 7 we briefly outline ongoing work and further directions to generalize techniques of this paper.

2. Preliminaries

We use the standard model-theoretic notion of relational structures. Let L be a language with relation symbols $R \in L$ each having its *arity*. An *L-structure* $\mathbf{A}$ on A is a structure with *vertex set* A and relations $R_\mathbf{A} \subseteq A^r$ for every symbol $R \in L$ of arity r. If the set A is finite, we call $\mathbf{A}$ a *finite structure*. We consider only structures with finitely many or countably infinitely many vertices.

Given two L-structures $\mathbf{A}$ and $\mathbf{B}$, a function $f\colon A \to B$ is an *embedding* $f\colon \mathbf{A} \to \mathbf{B}$ if it is injective and for every $R \in L$ of arity r we have that

$$(v_1, v_2, \ldots, v_r) \in R_\mathbf{A} \iff (f(v_1), f(v_2), \ldots, f(v_r)) \in R_\mathbf{B}.$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* if there is an embedding $f\colon \mathbf{A} \to \mathbf{B}$ that is onto.

As usual in the structural Ramsey theory, given an embedding $f\colon \mathbf{A} \to \mathbf{B}$ we will call the image of $\mathbf{A}$ in $\mathbf{B}$ (denoted by $f(\mathbf{A})$) a *copy* of $\mathbf{A}$ in $\mathbf{B}$. Structure $\mathbf{A}$ is *rigid* if the only automorphism of $\mathbf{A}$ (that is, isomorphism $\mathbf{A} \to \mathbf{A}$) is the identity. For rigid structures we will also slightly abuse the notation and write $\widetilde{\mathbf{A}} \in \binom{\mathbf{B}}{\mathbf{A}}$ for any structure for which there exists an embedding $f \in \binom{\mathbf{B}}{\mathbf{A}}$ such that $f(\mathbf{A}) = \widetilde{\mathbf{A}}$.

2.1. Parameter words and spaces.

Given a finite alphabet Σ and $k \in \omega \cup \{\omega\}$, a *$k$-parameter word* is a (possibly infinite) string W in alphabet $\Sigma \cup \{\lambda_i\colon 0 \leq i < k\}$ containing each of λ_i, $0 \leq i < k$, such that for every $1 \leq j < k$, the first occurrence of λ_j appears after the first occurrence of λ_{j-1}. Given a parameter word W, we denote by $|W|$ its *length* and for every $0 \leq j < |W|$ by W_j the letter (or parameter) on index j. (Note that the first letter of W has index 0). A 0-parameter word is simply a *word*. We will generally denote words by lowercase letters and parameter words by uppercase letters.

Let W be an n-parameter word and let U be a parameter word of length $k \leq n$ (where $k, n \in \omega \cup \{\omega\}$). Then we denote by $W(U)$ the parameter word created by *substituting* U to W. More precisely, this is a parameter word created from W by replacing each occurrence of λ_i, $0 \leq i < k$, by U_i and truncating it just before the first occurrence of λ_k (in W). Given an n-parameter word W and set S of parameter words of length at most n, we denote by $W(S)$ the set $\{W(U) \colon U \in S\}$.

We denote by $[\Sigma]\binom{n}{k}$ the set of all k-parameter words of length n (where $k \leq n \in \omega \cup \{\omega\}$). If k is finite we also denote by

$$[\Sigma]^*\binom{n}{k} = \bigcup_{i \leq n, i \in \omega} [\Sigma]\binom{i}{k}$$

the set of all finite k-parameter words of length at most n. For brevity we put $\Sigma^* = [\Sigma]^*\binom{\omega}{0}$, the set of all words on the alphabet Σ with finite length and no parameters. Given an n-parameter word W and integer $k < n$, we call $W([\Sigma]^*\binom{n}{k})$ the *k-dimensional subspace* described by W. We will denote by $\emptyset$ the empty word.

We will make use of the following infinitary variant of the Graham–Rothschild Theorem [GR71] which is a direct consequence of the Carlson–Simpson theorem [CS84]. This theorem was also obtained by Voigt around 1983 in a manuscript which, to our knowledge, was never published (see, i.e., [PV85, Theorem A], [Car87]),

Theorem 2.1. *Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set $[\Sigma]^*\binom{\omega}{k}$ is coloured by finitely many colours, then there exists an infinite-parameter word W such that $W\left([\Sigma]^*\binom{\omega}{k}\right)$ is monochromatic.*

We will also make use of the finite version of Theorem 2.1 (which follows from the Graham–Rothschild theorem by assigning every word $W \in [\Sigma]\binom{n+1}{k}$, $k > 1$, a word $W' \in [\Sigma]^*\binom{n}{k-1}$ created from W by truncating it just before the first occurrence of λ_{k-1}).

Theorem 2.2. *Let Σ be a finite alphabet, $0 \leq k \leq n$ and $r > 0$ finite integers. Then there exists $N = N(|\Sigma|, k, n, r)$ such that for every r-colouring of $[\Sigma]^*\binom{N}{k}$ there exists a word $W \in [\Sigma]^*\binom{N}{n}$ such that $W\left([\Sigma]^*\binom{n}{k}\right)$ is monochromatic.*

3. Envelopes and embedding types

Essentially all big Ramsey degree results are based on a notion of envelope and embedding type introduced by Laver and Milliken, see [Tod10, Section 6.2]. Precise definitions depend on the notion of a subspace (or a subtree). The following introduces these concepts for the context of parameter spaces.

Definition 3.1. Given a finite alphabet Σ and a set S of parameter words in alphabet Σ, an *envelope* of S is a parameter word W in alphabet Σ such that for every $U \in S$, there exists a parameter word U' such that $W(U') = U$. We call the envelope W *minimal* if there is no envelope of S with fewer parameters than W.

Example 1. Consider $\Sigma = \{0\}$ The set $S = \{0, 000\} \subseteq [\Sigma]^*\binom{\omega}{0}$ has two minimal envelopes: $0\lambda_0\lambda_0$ and $0\lambda_0 0$. Parameter word $\lambda_0\lambda_1\lambda_2\lambda_3$ is also an envelope of S, but it is not a minimal envelope.

Proposition 3.1. *Let Σ be a finite alphabet, let $k \geq 0$ be a finite integer, let S be a finite set of finite parameter words in alphabet Σ with at most k parameters and* *let W be a minimal envelope of S. Then W has at most $(|\Sigma| + k)^{|S|} + |S| - |\Sigma|$ parameters. Moreover, for every parameter λ_i of W and every minimal envelope W' of S it holds that the first occurrence of λ_i has the same position in W and W'.*

Proof. Fix Σ, k, and S. If the set S is empty, then W can be chosen to be the empty word. We thus assume that $|S| > 0$ and put $S = \{W^0, W^1, \dots, W^{\ell-1}\}$. We show a method to construct an envelope W.

Put $m = \max_{0 \leq i < \ell}(|W^i|)$ and for every $0 \leq i < m$ we define the *slice* i as a sequence $s^i = (W^0_i, W^1_i, \dots, W^{\ell-1}_i)$ where we put $W^j_i = *$ if $|W^j| \leq i$ (where $*$ is a special symbol not in Σ). For every $0 \leq i \leq j < m$ we say that slice i is *compatible* with slice j if for every $0 \leq p < \ell$ it holds that either $W^p_i = W^p_j$ or $W^p_j = *$ and $j \neq |W^p|$.

Now construct a word W of length m by putting for every $0 \leq j \leq m$

$$W_j = \begin{cases} s & \text{if slice } j \text{ is } (s, s, \dots, s) \text{ for some } s \in \Sigma, \\ W_{j'} & \text{if there exists } 0 \leq j' < j \text{ such that slice } j' \text{ is compatible with slice } j \\ & \text{and } j' \text{ is the minimal index with this property,} \\ \lambda_p & \text{otherwise, where } \lambda_p \text{ is the first so far unused parameter.} \end{cases}$$

It is easy to see that each new parameter must be introduced and thus the dimension of an envelope and first occurrences of parameters are uniquely determined by S. Since there are at most $(|\Sigma| + k)^\ell + \ell$ mutually incompatible slices, we will use at most $(|\Sigma| + k)^\ell + \ell - |\Sigma|$ parameters. □

Definition 3.2. Given a finite alphabet Σ, a finite integer $k \geq 0$, a set S of parameter words in alphabet Σ and an envelope W of S, an *embedding type* of S in W, denoted by $\tau_W(S)$, is the set of parameter words such that $W(\tau_W(S)) = S$.

Example 2. The set $S = \{0, 000\}$ has embedding type $\{\emptyset, 0\}$ in both minimal envelopes given in Example 1.

Corollary 3.2. *Let Σ be a finite alphabet and let $k, \ell > 0$ be finite integers. Then*

(1) the set

$$\{\tau_W(S) : S \subseteq [\Sigma]^* \binom{\omega}{k}, |S| = \ell, W \textit{ is a minimal envelope of } S\}$$

is finite, and,

(2) for every finite set $S \subseteq [\Sigma]^*\binom{\omega}{k}$ and its minimal envelopes W and W' it holds that $\tau_W(S) = \tau_{W'}(S)$.

As a consequence of Corollary 3.2 we can also use $\tau(S)$ for $\tau_W(S)$ where W is some minimal envelope of S.

Remark 3.1. Our Definitions 3.1 and 3.2 are closely related to the definition of envelopes and types used by Dodos, Kanellopoulos and Tyros [DKT14] and by Furstenberg and Katznelson [FK89], see also [DK16, Chapter 5]. The main difference is however the use of subspaces defined by variable words rather than parameter words. With respect to this notion subspaces the dimension of envelope and thus also the number of types is not bounded by the size of the set.

4. Big Ramsey degrees

In this section we prove Theorems 1.1 and 1.2. We start with Theorem 1.2 and later show that Theorem 1.1 follows by very similar arguments.

4.1. **Triangle-free graphs.** In this section we consider graphs to be structures in a language consisting of a single binary relation E. We fix alphabet $\Sigma = \{0\}$.

Definition 4.1. We define graph $\mathbf{G}$ as follows:

(1) The vertex set G is $[\Sigma]^*\binom{\omega}{1}$ (that is, the set of all finite 1-parameter words).
(2) Given two vertices U and V such that $|U| < |V|$, we put an edge between U and V if and only if
  (i) $V_{|U|} = \lambda_0$ and
  (ii) for no $0 \leq j < |U|$ it holds that $U_j = V_j = \lambda_0$.

There are no other edges.

Remark 4.1. Condition (i) in Definition 4.1 is the passing number representation of the Rado graph used by Sauer [Sau06] (see also [Tod10, Theorem 6.25]). Condition (ii) is similar to Dobrinen's parallel 1's criterion [Dob20a, Definition 3.7]. The notion of subtree (or a subspace) used here is however different from [Sau06] and [Dob20a].

Lemma 4.1. *Graph* $\mathbf{G}$ *is triangle-free.*

Proof. Assume to the contrary that U, V and W form a triangle. Without loss of generality we can assume that $|U| < |V| < |W|$. Because there is an edge between U and V, we know that $V_{|U|} = \lambda_0$. Because there is an edge between U and W, we know that $W_{|U|} = \lambda_0$. A contradiction with the existence of an edge between V and W. □

The following follows directly from the definition of the substitution:

Observation 4.2. *Let W be an infinite-parameter word. Then for every $U, V \in G$ it holds that U is adjacent to V if and only if $W(U)$ is adjacent to $W(V)$.*

Let $\mathbf{H}$ with $H = \omega$ be (an enumeration of) the universal homogeneous triangle-free graph. We define the mapping $\varphi\colon \omega \to G$ by putting $\varphi(i) = U$ where U is a 1-parameter word of length i defined by putting for every $0 \leq j < i$

$$U_j = \begin{cases} \lambda_0 & \text{if and only if } \{j, i\} \text{ is an edge of } \mathbf{H}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check:

Observation 4.3. *Function φ is an embedding $\varphi\colon \mathbf{H} \to \mathbf{G}$ and thus $\mathbf{G}$ is a universal triangle-free graph.*

Now we prove Theorem 1.2 in the following form:

Theorem 4.4. *For every finite $k \geq 1$ and every finite colouring of induced subgraphs of $\mathbf{G}$ with k vertices there exists $f \in \binom{\mathbf{G}}{\mathbf{G}}$ such that the colour of every k-vertex subgraph $\mathbf{A}$ of $f(\mathbf{G})$ depends only on $\tau([A]) = \tau(f^{-1}[A])$.*

Observe that by Corollary 3.2, we obtain the desired finite upper bound on number of colours. The proof is again structured similarly to Milliken and Laver's results, see [Tod10, Section 6.3]: by a repeated application of Theorem 2.1, we obtain the desired copy.

Proof. Fix k and a finite colouring χ of subsets of G of size k. Let $T^0, T^1, \ldots, T^{N-1}$ be all possible embedding types of subsets of G of size k in their minimal envelopes (given by Corollary 3.2). For every $0 \leq i \leq N-1$, put $n_i = \max\{|U|\colon U \in T^i\}$.

Choose an infinite-parameter word $W^0 \in [\Sigma]\binom{\omega}{\omega}$ arbitrarily. We construct a sequence of infinite-parameter words $W^1, W^2, \ldots, W^N$ such that for every $0 < i \leq N$ the following is satisfied:

(1) $W^i = W^{i-1}(Z^i)$ for some infinite-parameter word Z^i,
(2) There exists colour c^i such that
$$\chi(W^i(U(T^{i-1}))) = c^i$$
for every $U \in [\Sigma]^* \binom{\omega}{n_{i-1}}$.

Let f be defined by $f(U) = W^N(U)$. By Observation 4.2 we know that this is an embedding with the desired properties.

It remains to show the construction of W^i. Assume that W^{i-1} is constructed. Let $\chi^i\colon [\Sigma]^* \binom{\omega}{n_{i-1}}$ be a colouring given by $\chi^i(U) = \chi(W^{i-1}(U(T^{i-1})))$. By Theorem 2.1 there exists an infinite-parameter word Z^i and colour c^i satisfying that $\chi^i(Z^i(U)) = c^i$ for every $U \in [\Sigma]^* \binom{\omega}{n_{i-1}}$. Put $W^i = W^{i-1}(Z^i)$. □

4.2. **Partial orders.** Throughout this section we fix a language with a single binary relation $\leq$ and consider a partial order $(A, \leq_{\mathbf{A}})$ to be a structure $\mathbf{A}$ with vertex set A and a binary relation $\leq_{\mathbf{A}}$. We also fix the alphabet $\Sigma = \{L, X, R\}$. We will use the lexicographic order of words that is based on the following order of the alphabet: $L <_{\mathrm{lex}} X <_{\mathrm{lex}} R$. We define the following binary relation on Σ^*:

Definition 4.2. For $w, w' \in \Sigma^*$ we put $w \prec w'$ if and only if there exists $0 \leq i < \min(|w|, |w'|)$ such that
(i) $(w_i, w'_i) = (L, R)$ and
(ii) for every $0 \leq j < i$ it holds that $w_j \leq_{\mathrm{lex}} w'_j$.

For $w \prec w'$ we denote by $i(w, w')$ the minimal i satisfying the condition (i) above. We put $w \preceq w'$ if and only if either $w = w'$ or $w \prec w'$.

We denote by $\mathbf{O}$ the structure with vertex set $O = \Sigma^*$ ordered by $\preceq$. (Thus we put $u \leq_{\mathbf{O}} v$ if and only if $u \preceq v$.)

Proposition 4.5. *Structure* $\mathbf{O}$ *is a partial order.*

The intuitive meaning of the definition above is that for every $w \in \Sigma^*$ and every j the letter w_j describes a position of the vertex w with respect to an extension of the partial order by a new vertex. When extending a given partial order by a new vertex v, we obtain a partitioning of its vertex set into three sets: L is the set of all vertices smaller than v, X is the set of all vertices not comparable to v and R is the set of all vertices greater than v. Because we aim to define partial order on the set of all words and because not every choice of L, X and R represent an extension, we simply disregard all the information which is in conflict with what has been decided earlier.

Proof of Proposition 4.5. It is easy to see that $\preceq$ is reflexive and anti-symmetric. We verify transitivity. Let $w \prec w' \prec w''$ and put $i = \min(i(w, w'), i(w', w''))$.

First assume that $i = i(w, w')$. Then we have $w_i = L, w'_i = R$ which implies that $w''_i = R$. For every $0 \leq j < i$ it holds that $w_j \leq_{\mathrm{lex}} w'_j \leq_{\mathrm{lex}} w''_j$. It follows that $w \preceq w''$ and $i(w, w'') \leq i$.

Now assume that $i = i(w', w'')$. Then we have $w'_i = L$, $w''_i = R$ and because $w'_i = L$ then also $w_i = L$. Again for every $0 \leq j < i$ it holds that $w_j \leq_{\mathrm{lex}} w'_j \leq_{\mathrm{lex}} w''_j$. It also follows that $w \preceq w''$ and $i(w, w'') \leq i$. □

The key to our construction is the following:

Lemma 4.6. *Let* W *be an infinite-parameter word. Then for every* $w, w' \in \Sigma^*$ *it holds that* $w \preceq w'$ *if and only if* $W(w) \preceq W(w')$.

Proof. This can be easily checked using the fact that for every $i > 0$, λ_i first occurs in W after the first occurrence of λ_{i-1}. □

Recall that by $\mathbf{P} = (P, \leq_{\mathbf{P}})$ we denote the universal homogeneous partial order. Without loss of generality, we can assume that $P = \omega$ and thus fix an (arbitrary) enumeration of $\mathbf{P}$. We define function $\varphi\colon \omega \to \Sigma^*$ by mapping $j \in P$ to a word w of length $2j$ defined as:

$$(w_{2i}, w_{2i+1}) = \begin{cases} (L, L) & \text{for every } i < j,\ j \leq_{\mathbf{P}} i, \\ (R, R) & \text{for every } i < j,\ i \leq_{\mathbf{P}} j, \\ (X, X) & \text{for every } i < j,\ i \text{ is incomparable with } j \text{ by } \leq_{\mathbf{P}}, \\ (L, R) & \text{for } i = j. \end{cases}$$

Proposition 4.7. *The function φ is an embedding $\varphi\colon \mathbf{P} \to \mathbf{O}$. Consequently, $\mathbf{O}$ is a universal partial order.*

Proof. Given $i < j \in \omega$, put $u = \varphi(i)$ and $v = \varphi(j)$ and consider three cases:

1. $i \leq_{\mathbf{P}} j \implies u \preceq v$: We have $u_{2i} = L$ and $v_{2i} = R$ and we check that for every $0 \leq k < i$ it holds that $u_{2k} \leq_{\text{lex}} v_{2k}$ and thus also $u_{2k+1} \leq_{\text{lex}} v_{2k+1}$. If $u_{2k} = L$ then this follows trivially. If $u_{2k} = X$ then we know that k is incomparable with i by $\leq_{\mathbf{P}}$. It follows that $v_{2k} \neq L$ because $i \leq_{\mathbf{P}} j$ and thus it can not hold that $j \leq_{\mathbf{P}} k$. If $u_{2k} = R$ then we get $k \leq_{\mathbf{P}} i \leq_{\mathbf{P}} j$ and thus also $v_{2k} = R$.
2. $j \leq_{\mathbf{P}} i \implies v \preceq i$: Here we have $u_{2i+1} = R$ and $v_{2i+1} = L$. Analogously as in the previous case we can check that for every $0 \leq k < i$ it holds that $v_{2k} \leq_{\text{lex}} u_{2k}$.
3. If i is incomparable with j in $\leq_{\mathbf{P}}$ then u is incomparable with v in $\preceq$: Assume the contrary and let $k \leq i$ be such that either $u_{2k} = L$ and $v_{2k} = R$ or $u_{2k+1} = L$ and $v_{2k+1} = R$. Clearly $k < i$ because $v_{2i} = v_{2i+1} = X$. We get that $i \leq_{\mathbf{P}} k \leq_{\mathbf{P}} j$. A contradiction.

□

Remark 4.2. Easy constructions of universal partial orders are interesting in their own right, see [Hed69, PT80, HN05b, HN05a, HN11]. Observe also that $\leq_{lex}$ is a linear extension of $\preceq$ and thus the construction can be seen as a direct refinement of the Laver–Devlin construction.

Now we are ready to prove Theorem 1.1 in the following form.

Theorem 4.8. *For every finite $k \geq 1$ and every finite colouring of (induced) suborders of $\mathbf{O}$ with k elements, there exists $f \in \binom{\mathbf{O}}{\mathbf{O}}$ such that the colour of every suborder $\mathbf{A}$ of $f(\mathbf{O})$ with k vertices depends only on $\tau([A]) = \tau(f^{-1}[A])$.*

Proof. This follows in an analogy to Theorem 4.4.

Fix k and a finite colouring χ of subsets of O of size k. Let $T^0, T^1, \dots, T^{N-1}$ be all possible embedding types of subsets of O of size k in their minimal envelopes (given by Corollary 3.2). For every $0 \leq i \leq N-1$ put $n_i = \max\{|U|\colon U \in T^i\}$.

Choose infinite-parameter word $W^0 \in [\Sigma]\binom{\omega}{\omega}$ arbitrarily. We construct a sequence of infinite-parameter words $W^1, W^2, \dots, W^N$ such that for every $0 < i \leq N$ the following is satisfied:

1. $W^i = W^{i-1}(Z^i)$ for some infinite-parameter word Z^i,
2. There exists colour c^i such that
$$\chi(W^i(U(T^{i-1}))) = c^i$$
for every $U \in [\Sigma]^*\binom{\omega}{n_{i-1}}$.

Let f be defined by $f(U) = W^N(U)$. By Lemma 4.6 we know that this is an embedding with the desired properties.

Word W^i is again constructed by an application of Theorem 2.1. □

5. Ramsey classes (of finite structures)

5.1. **Ordered triangle-free graphs.** An *ordered graph* is a relational structure $\mathbf{A}$ in a language consisting of two binary relations E and $\leq$ such that $(A, E_\mathbf{A})$ is a graph and $(A, \leq_\mathbf{A})$ is a linear order.

We prove a special case of the Nešetřil–Rödl theorem. Our proof is based on the ideas developed in the previous sections and is arguably the most direct proof of this result known to date, giving a particularly simple description of the Ramsey graph $\mathbf{C}$. We shall remark that similar constructions have been known for unrestricted classes, see [Prö13, Theorem 12.13] for a proof of the Ramsey property of the class of all finite ordered graphs. However, to our best knowledge, a similar strategy has been applied to a class of graphs with a forbidden subgraph in a special cases only (for coloring vertices and edges [NR75a, NR75b]).

Theorem 5.1 (Nešetřil–Rödl)**.** *For every integer $r > 0$ and every pair of finite ordered triangle-free graphs $\mathbf{A}$ and $\mathbf{B}$, there exists a finite ordered triangle-free graph $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^\mathbf{A}_{r,1}$.*

Proof. We fix alphabet $\Sigma = \{0\}$. Recall the graph $\mathbf{G}$ defined in Definition 4.1. By $\mathbf{G}_N$, we denote the ordered graph created from $\mathbf{G}$ by considering only vertices in $[\Sigma]^*\binom{N}{1}$ and adding a lexicographic ordering of the vertices (where we consider vertices to be strings in alphabet $\{0, \lambda_0\}$ ordered $0 \leq_{\text{lex}} \lambda_0$).

We will show that for sufficiently large N (to be specified at the end of the proof) it holds that $\mathbf{G}_N \longrightarrow (\mathbf{B})^\mathbf{A}_{r,1}$. Towards this, we first define a more careful way to embed an ordered triangle-free graph $\mathbf{B}$ to a graph $\mathbf{G}_n$.

Let $\mathbf{B}$ be an ordered triangle-free graph. For simplicity we can assume that $B = \{0, 1, \dots, |B|-1\}$ and that $\leq_\mathbf{B}$ coincides with the order of integers. We define an embedding $\varphi\colon \mathbf{B} \to \mathbf{G}_n$ for some sufficiently large n to be fixed later by the following procedure. We say that a function $f\colon B \to \{0, \lambda_0\}$ is a *Katětov function* (for $\mathbf{B}$) if $\mathbf{B}$ extended by a new vertex which is adjacent precisely to those vertices $v \in B$ satisfying $f(v) = \lambda_0$ is a triangle free graph. In other words, there are no two adjacent vertices $v, v' \in B$ such that $f(v) = f(v') = \lambda_0$.

Now enumerate all possible Katětov functions as $f_0, f_1, \dots, f_{d-1}$ ordered lexicographically with respect to $\leq_\mathbf{B}$. More precisely, we see every function f_i as a word w^i of length $|B|$ with $w^i_j = f_i(j)$ and order those words lexicographically.

Put $\varphi(v) = V$ where $|V| = d + v$ and

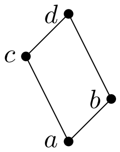
$$V_j = \begin{cases} f_j(v) & \text{for } j < d, \\ \lambda_0 & \text{for } d \leq j < d + v \text{ such that } v \text{ is adjacent to } j - d \text{ in } \mathbf{B}, \\ 0 & \text{for } d \leq j < d + v \text{ such that } v \text{ is not adjacent to } j - d \text{ in } \mathbf{B}. \end{cases}$$

Now put $n = d + |B|$. It is easy to see that φ is an embedding of $\mathbf{B}$ to $\mathbf{G}_n$ (to see that the order is preserved, note that all extensions by a vertex connected to precisely one vertex of $\mathbf{B}$ are triangle-free). An example of this representation is depicted in Figure 1.

Let φ' be an embedding of $\mathbf{A} \to \mathbf{G}_k$ for some $k > 0$ constructed in the same way as above.

Claim 5.2. *For every $\widetilde{\mathbf{A}} \in \binom{\mathbf{B}}{\mathbf{A}}$ there exists a k-parameter word $W \in [\Sigma]^*\binom{n}{k}$ such that $W(\varphi'(A)) = \varphi(\widetilde{A})$.*

Let $f_0, f_1, \dots, f_{d-1}$ be the enumeration of Katětov functions of $\mathbf{B}$ in the lexicographic order and let $f'_1, f'_2, \dots, f'_{d'-1}$ be the enumeration of Katětov functions of $\widetilde{\mathbf{A}}$ also ordered lexicographically. Let $h\colon \{0, 1, \dots, d-1\} \to \{0, 1, \dots, d'-1\}$ be the



$\varphi(d) = 0\lambda0000\lambda0\lambda\lambda$

$\varphi(c) = 00\lambda0\lambda00\lambda0$

$\varphi(b) = 000\lambda\lambda00\lambda$

$\varphi(a) = 00000\lambda\lambda$

f • $\varphi'(f) = 0\lambda0\lambda0$

e • $\varphi'(e) = 00\lambda\lambda$

$e \mapsto b, f \mapsto c : \lambda_0\lambda_0\lambda_1\lambda_2\lambda_3\lambda_0\lambda_0\lambda_3\lambda_4$

$e \mapsto a, f \mapsto d : \lambda_0\lambda_1\lambda_0\lambda_0\lambda_0\lambda_2\lambda_3\lambda_4\lambda_0\lambda_1\lambda_1$

FIGURE 1. Representation of a graph $\mathbf{B}$ ordered $a \leq_{\mathbf{B}} b \leq_{\mathbf{B}} c \leq_{\mathbf{B}} d$ and a graph $\mathbf{A}$ ordered $e \leq_{\mathbf{A}} f$ along with a parameter word representing all embeddings of $\mathbf{A}$ to $\mathbf{B}$ as constructed in the proof of Claim 5.2. For easier reading, λ_0 is typeset as λ.

mapping such that for every $i \in \{0, 1, \dots, d-1\}$ function f_i restricted to $\widetilde{A}$ is $f'_{h(i)}$. Observe that every Katětov function f of $\widetilde{\mathbf{A}}$ can be extended to a Katětov function f' of $\mathbf{B}$ by putting $f' = f(v)$ for $v \in \widetilde{A}$ and $f'(v) = 0$ otherwise. It follows that h exists and is surjective.

Let θ be the isomorphism $\mathbf{A} \to \widetilde{\mathbf{A}}$. For every $v \in B \setminus \widetilde{A}$ we put $e(v)$ to be an integer such that $f'_{e(v)}$ describes the neighbourhood of v in $\widetilde{A}$.

We now define a string W of length $d + \max(\widetilde{A})$ as follows:

$$W_j = \begin{cases} \lambda_{h(j)} & \text{for every } 0 \leq j < d, \\ \lambda_{d'+\theta^{-1}(j-d)} & \text{for every } d \leq j \text{ such that } j - d \in \widetilde{A}, \\ \lambda_{e(j-d)} & \text{for every } d \leq j \text{ such that } j - d \notin \widetilde{A}. \end{cases}$$

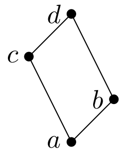
First observe that $W_0 = \lambda_0$. This is because f_0 and f'_0 are both constant zero functions.

We verify that W is a k-parameter word, that is, for every $1 \leq j < k$ it holds that the first occurrence of λ_j comes after the first occurrence of λ_{j-1}. We consider three cases:

(1) $j < d'$: Function f'_j can be extended to function $f''_j \colon B \to \{0, \lambda_0\}$ by putting $f''_j(v) = 0$ for every $v \notin \widetilde{A}$. This is clearly a Katětov function of $\mathbf{B}$ and therefore there exists j' such that $f''_j = f_{j'}$. From this it follows that $W_{j'} = \lambda_j$. Because zero is the minimal element of the alphabet we get that this is also the first occurrence of λ_j in W. Finally because the first occurrence of λ_{j-1} can be found same way and the extension by zeros preserves the relative lexicographic order, we know that λ_j appears after λ_{j-1}.
(2) $j = d'$: λ_j occurs once at position $d + \theta(j - d') = d + \theta(0)$. We already checked that λ_{j-1} occurs before d.
(3) $d' < j < k$: For every $d' < j < k$ it holds that λ_j occurs precisely once at position $d + \theta(j - d')$ so the desired ordering follows form the monotonicity of θ.

This finishes the proof that W is indeed k-parameter word. By substituting $\varphi'(A)$ into W it can be also checked that $W(\varphi'(A)) = \varphi(\widetilde{A})$. This finishes the proof of Claim 5.2.

Now let $N = N(1, n, k, r)$ be given by Theorem 2.2. We claim that $\mathbf{G}_N \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,1}$. Consider an r-colouring of $\mathbf{G}_N$. Observe that for every $W \in [\Sigma]^*\binom{N}{k}$ we get a unique copy of $\mathbf{A}$ in $\mathbf{G}_N$ given by $W(\varphi'(A))$. We thus obtain an r-colouring of $[\Sigma]^*\binom{N}{k}$ and by an application of Theorem 2.2 a word $\widetilde{W} \in [\Sigma]^*\binom{N}{n}$ for which this colouring is constant. The monochromatic copy of $\mathbf{B}$ is now given by $\widetilde{W}(\varphi(B))$. $\square$



$\varphi(d) = \texttt{LXXXXXXRRRL}$

$\varphi(c) = \texttt{LLXLXXRRL}$

$\varphi(b) = \texttt{LLLXXXRL}$

$\varphi(a) = \texttt{LLLLLXL}$

$\varphi'(f) = \texttt{LXLXRL}$

$\varphi'(e) = \texttt{LLXXL}$

$e \mapsto b, f \mapsto c : \lambda_0\lambda_0\lambda_1\lambda_2\lambda_3\lambda_3\texttt{RR}\lambda_5\lambda_6$

FIGURE 2. Representation of a partial order $\mathbf{B}$ with linear extension $a \leq_{\mathbf{B}} b \leq_{\mathbf{B}} c \leq_{\mathbf{B}} d$ and a partial order $\mathbf{A}$ with linear extension $a \leq_{\mathbf{A}} b$ (relations $\trianglelefteq_{\mathbf{B}}$ and $\trianglelefteq_{\mathbf{A}}$ are depicted by Hasse diagrams) along with a parameter word representing the embedding of $\mathbf{A}$ to $\mathbf{B}$ as constructed in the proof of Claim 5.4.

5.2. **Partial orders with linear extension.** Now we will consider structures in language with two binary relations $\leq$ and $\trianglelefteq$. $\mathbf{A}$ is a *partial order with linear extension* if $(A, \trianglelefteq_{\mathbf{A}})$ a partial order and $(A, \leq_{\mathbf{A}})$ its linear extension.

We prove:

Theorem 5.3 ([NR84, PTW85])**.** *For every integer $r > 0$ and every pair of finite partial orders with linear extensions $\mathbf{A}$ and $\mathbf{B}$ there exists a finite partial order with linear extension $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,1}$.*

Remark 5.1. The proof of Theorem 5.3 presented here is related to proofs of this result based on the Graham–Rothschild theorem (by Fouché [Fou97], see also [Maš18, Theorem 4.1]). We present it because our representation of the partial order by finite words is different. This difference is necessary to show Theorem 1.1 (where countably infinite partial orders need to be represented), but also perhaps makes the proof of Theorem 5.3 a bit more systematic.

Proof. We fix alphabet $\Sigma = \{L, X, R\}$ and its ordering $L <_{\text{lex}} X <_{\text{lex}} R$. Denote by $\mathbf{O}_N$ the partial order induced on $[\Sigma]^*\binom{N}{0}$ by $\mathbf{O}$ (given by Definition 4.2) with a linear extension defined by the lexicographic order.

Fix $\mathbf{A}$ and $\mathbf{B}$ and proceed in analogy to the proof of Theorem 5.1. For simplicity we can assume that $B = \{0, 1, \dots, |B| - 1\}$ and that $\leq_{\mathbf{B}}$ coincides with the order of integers. We show that there exists N such that $\mathbf{O}_N \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,1}$.

We define an embedding $\varphi\colon \mathbf{B} \to \mathbf{O}_n$ for some sufficiently large n (to be fixed later) by the following procedure. We say that function $f\colon B \to \{L, X\}$ *represents a downset* of $\mathbf{B}$ if the set $\{v\colon f(v) = L\}$ is downwards closed with respect to $\trianglelefteq_{\mathbf{B}}$.

Now enumerate all possible functions representing a downset as $f_0, f_1, \dots, f_{d-1}$ ordered lexicographically with respect to $\leq_{\mathbf{B}}$. Put $\varphi(v) = w$ where is a word of length $d + v + 1$ defined as follows:

$$w_j = \begin{cases} f_j(v) & \text{for } 0 \leq j < d, \\ R & \text{for } d \leq j < d + v, \\ L & \text{for } j = d + v. \end{cases}$$

An example of this representation is depicted in Figure 2.

Now put $n = d + |B| + 1$. It is easy to see that φ is an embedding of $\mathbf{B}$ to $\mathbf{O}_n$: levels d to $d + |B|$ code the linear extensions given by $\leq_{\mathbf{B}}$ while earlier levels code all downsets. Every pair of vertices $u \leq_{\mathbf{B}} v$ which are not comparable by $\trianglelefteq$ have downsets witnessing this which makes sure that their images are also not comparable by $\preceq$.

Let φ' be an embedding of $\mathbf{A} \to \mathbf{O}_k$ for some $k > 0$ constructed the same way as above.

Claim 5.4. *For every $\widetilde{\mathbf{A}} \in \binom{\mathbf{B}}{\mathbf{A}}$ there exists a k-parameter word $W \in [\Sigma]^*\binom{n}{k}$ such that $W(\varphi'(A)) = \varphi(\widetilde{A})$.*

Let $f_0, f_1, \dots, f_{d-1}$ be the enumeration of functions representing downsets of $\mathbf{B}$ in the lexicographic order (with respect to $\leq_{\mathbf{B}}$) and $f'_1, f'_2, \dots, f'_{d'-1}$ be the enumeration of functions representing downsets of $\widetilde{\mathbf{A}}$ also ordered lexicographically. Let $h\colon \{0, 1, \dots, d-1\} \to \{0, 1, \dots, d'-1\}$ be the mapping such that f_i restricted to $\widetilde{A}$ is $f'_{h(i)}$. Observe that every downset f of $\widetilde{\mathbf{A}}$ can be extended to a downset of f' and thus h is well defined and surjective.

Let θ be the embedding $\mathbf{A} \to \widetilde{\mathbf{A}}$. We now define a string W of length $d+\max(\widetilde{A})$ as follows:

$$W_j = \begin{cases} \lambda_{h(j)} & \text{for every } 0 \leq j < d, \\ \lambda_{d'+\theta^{-1}(j-d)} & \text{for every } d \leq j < |B|+1 \text{ such that } j-d \in \widetilde{A}, \\ R & \text{for every } d \leq j < |B|+1 \text{ such that } j-d \notin \widetilde{A}. \end{cases}$$

Next we verify that W is a k-parameter word. For this we need to find for every f'_j its lexicographically minimal extension $f_{j'}$ and verify that the lexicographic order is preserved. Given f'_j, we construct function $f\colon \mathbf{B} \to \{L, X, R\}$ by putting:

$$f(v) = \begin{cases} f'_j(v) & \text{if } v \in \widetilde{A}, \\ X & \text{if } v \notin \widetilde{A} \text{ and there exists } u \in \widetilde{A},\ f'_j(u) = X \text{ and } u \trianglelefteq_{\mathbf{B}} v, \\ L & \text{otherwise.} \end{cases}$$

Observe that there is j' such that $f = f_{j'}$ and that $f_{j'}$ is lexicographically minimal among all functions f_ℓ which represent a downset of $\mathbf{B}$ such that $h(f_\ell) = f'_j$. This is due to fact that we put $f(v) = X$ only when this was forced by a "witness" $u \in \widetilde{A}$ for which $f'_j(u) = X$, and thus the value of v is X in every extension of f'_j which represents a downset. To see that this construction preserves the lexicographic order, it remains to observe that since $u \trianglelefteq_{\mathbf{B}} v$, we also have $u \leq_{\mathbf{B}} v$ and thus while constructing the lexicographic order of extensions, $f(u)$ will take a precedence over $f(v)$.

Note that at this moment we make use of the fact that our representation uses downsets rather than all Katětov functions which would seem as more direct analogy of the proof of Theorem 5.1.

We thus conclude that W is indeed a parameter word. This finishes the proof of the claim.

Now let $N = N(0, n, k, r)$ be given by Theorem 2.2. We claim that $\mathbf{O}_N \longrightarrow (\mathbf{B})^{\mathbf{A}}_{r,1}$. Consider an r-colouring of $\mathbf{O}_N$. Observe that for every $W \in [\Sigma]^*\binom{N}{k}$ we get a unique copy of $\mathbf{A}$ in $\mathbf{O}_N$ given by $W(\varphi'(A))$. We thus obtain an r-colouring of $[\Sigma]^*\binom{N}{k}$ and by application of Theorem 2.2 a word $\widetilde{W} \in [\Sigma]^*\binom{N}{n}$ for which this colouring is constant. The monochromatic copy of $\mathbf{B}$ is now given by $\widetilde{W}(\varphi(B))$. $\square$

6. Applications

In this section we briefly discuss some examples of structures where finiteness of big Ramsey degrees follows as a direct consequence of Theorems 1.1 and 4.8. This includes some already known examples (linear orders, graphs, triangle-free graphs, ultrametric spaces) as well and a new example (S-metric spaces).

For each of the examples we will construct an interpretation in the universal homogeneous partial order $\mathbf{P}$ (or its fixed linear extension) which has the property that vertices of this interpretation are formed by $\binom{\mathbf{P}}{\mathbf{V}}$ for some finite poset $\mathbf{V}$. By obtaining a common representation of these structures within partial orders we also show that free superpositions of such structures have finite big Ramsey degrees, thereby giving a partial answer to a question asked by Zucker during the 2018 BIRS workshop "Unifying Themes in Ramsey Theory."

We stress that the representations here generally only lead to very generous upper bounds on big Ramsey degrees.

6.1. **Triangle-free graphs.** It may be a bit of a surprise that Theorem 1.1 implies Theorem 1.2 in a particularly easy way. Given a homogeneous partial order $\mathbf{P}$, we denote by $\mathbf{G_P}$ the following graph:

(1) Vertices of $\mathbf{G_P}$ are all triples of distinct vertices (u_0, u_1, u_2) of $\mathbf{P}$ such that $u_0 <_\mathbf{P} u_2$, while (u_0, u_1) and (u_1, u_2) are incomparable in $\mathbf{P}$.
(2) Vertices (u_0, u_1, u_2) and (v_0, v_1, v_2) form an edge of $\mathbf{G_P}$ if and only if $u_0 <_\mathbf{P} v_1 <_\mathbf{P} u_2$, $v_0 <_\mathbf{P} u_1 <_\mathbf{P} v_2$ and all other pairs (u_i, v_j), $i, j \in \{0, 1, 2\}$, are incomparable in $\mathbf{P}$.

By transitivity, $\mathbf{G_P}$ is triangle-free: if both $\{(u_0, u_1, u_2), (v_0, v_1, v_2)\}$ and $\{(v_0, v_1, v_2), (w_0, w_1, w_2)\}$ are edges of $\mathbf{G_P}$ then we have $u_0 \leq_\mathbf{P} w_2$ which implies that $\{(u_0, u_1, u_2), (w_0, w_1, w_2)\}$ is a non-edge.

It is not hard to check that there is an embedding φ form the homogeneous universal triangle free graph $\mathbf{H}$ to $\mathbf{G_P}$. Recall that the vertex set of $\mathbf{H}$ is ω and construct the embedding φ inductively. For each vertex $i \in \omega$ assume that $\varphi(i')$ is constructed for every $i' < i$ and apply the extension property of $\mathbf{P}$ to obtain three disjoint vertices $i_0, i_1, i_2 \in P$, such (i_0, i_1, i_2) is a vertex of $\mathbf{G_P}$, adn for every $j \leq i$ vertices $\varphi(j) = (j_0, j_1, j_2)$ are disjoint from (i_0, i_1, i_2) and the following is satisfied:

(1) If i, j forms an edge of $\mathbf{H}$ put $i_0 \leq_\mathbf{P} j_1 \leq_\mathbf{P} i_2$ and $j_0, \leq_\mathbf{P} i_1, \leq_\mathbf{P} j_2$ so (i_0, i_1, i_2) and (j_0, j_1, j_2) forms an edge of $\mathbf{G_P}$.
(2) If i, j does not form an edge of $\mathbf{H}$ put $i_0 \leq_\mathbf{P} j_2$ and $j_0 \leq_\mathbf{P} i_2$ while keeping all other pairs (i_k, j'_k), $k \in \{0, 1, 2\}$ incomparable in $\mathbf{P}$.

To finish the proof of Theorem 1.2, assume that we are given a finite colouring of $\binom{\mathbf{H}}{\mathbf{A}}$ for some finite triangle-free graph $\mathbf{A}$. Since $\mathbf{H}$ is universal, it contains a copy of $\mathbf{G_P}$ and hence it induces a colouring of $\binom{\mathbf{G_P}}{\mathbf{A}}$. This can be turned into a finite colouring of substructures of $\mathbf{P}$ on at most $3|A|$ vertices and hence, by Theorem 1.1, there is a copy of $\mathbf{P}$ with at most a bounded number of colours. This corresponds to a copy of $\mathbf{G_P}$ in $\mathbf{G_P}$ with at most a bounded number of colours and the rest follows by universality of $\mathbf{G_P}$.

6.2. **Urysohn S-metric spaces.** Let S be a set of non-negative reals such that $0 \in S$. A metric space $\mathbf{M} = (M, d)$ is an *S-metric space* if for every $u, v \in M$ it holds that $d(u, v) \in S$. We call a countable S-metric space $\mathbf{M}$ a *Urysohn S-metric space* if it is homogeneous (that is, every isometry of its finite subspaces extends to a bijective isometry from $\mathbf{M}$ to $\mathbf{M}$) and every countable S-metric space embeds to it. In the following we will see S-metric spaces as relational structures in a language with a binary relation R_ℓ for every $\ell \in S \setminus \{0\}$.

Finite set of non-negative reals $S = \{0 = s_0 < s_1 < \cdots < s_n\}$ is *tight* if if $s_{i+j} \leq s_i + s_j$ for all $0 \leq i \leq j \leq i + j \leq n$ (see [Maš18]). It follows from a classification by Sauer [Sau13] that for every such S there exists an Urysohn S-metric space.

Mašulović in [Maš18, Theorem 4.4] shows a way to represent all S-metric cases with finitely many distances (for every tight set S) as a partial order. Using this construction we obtain:

Corollary 6.1. *Let S be a finite tight set of non-negative reals. Then the Urysohn S-metric space has finite big Ramsey degrees.*

We will show a special case of Corollary 6.1 where $S = \{0, 1, \dots, d\}$. For other tight sets we refer the reader to [Maš18, Theorem 4.4].

Proof (sketch). Fix d and $S = \{0, 1, \dots, d\}$. Construct an S-metric space $\mathbf{M}_S$ as follows:

(1) Vertices are chains of vertices of $\mathbf{P}$ of length d.
(2) Given two chains $u_0 \leq_{\mathbf{P}} \cdots \leq_{\mathbf{P}} u_{d-1}$ and $v_0 \leq_{\mathbf{P}} \cdots \leq_{\mathbf{P}} v_{d-1}$ their distance is the minimal $\ell \in \{0, 1, \dots d\}$ such that for every $i \in \{0, 1, \dots d - i\}$ it holds that $u_i \leq_{\mathbf{P}} v_{i+\ell}$ and $v_i \leq_{\mathbf{P}} u_{i+\ell}$.

Just as in the case of triangle-free graphs, triangle inequality follows from transitivity, one can embed the Urysohn S-metric space to $\mathbf{M}_S$ using an on-line algorithm and hence Corollary 6.1 follows. □

Note that not all finite sets S for which there exists a Urysohn S-metric space (these were characterised by Sauer [Sau13]) are tight and thus Corollary 6.1 is not a complete characterisation.

Just like Theorem 1.1, Corollary 6.1 has a known finite form. Ramsey property of the class of all finite ordered metric spaces was shown by Nešetřil [Neš07] (see also [DR12] for graph metric spaces). This result was later generalised to all S-metric spaces [HN19, HKN19a]. Big Ramsey degrees of vertices for all Urysohn S-metric spaces with S finite was shown to be one by Sauer [Sau12]. See also [DLPS07, DLPS08, NVT10] for more background on vertex partition theorems of Urysohn spaces.

6.3. **Ultrametric spaces.** Recall that metric space $\mathbf{M} = (M, d)$ is an *ultrametric space* if the triangle inequality can be strengthened to $d(u, w) \leq \max\{d(u, v), d(v, w)\}$. The *Urysohn ultrametric space of diameter d* is the universal and homogeneous ultrametric space with distances $\{0, 1, \dots, d\}$. The following was shown by Nguyen Van Thé [NVT09] (along with a full characterisation of big Ramsey degrees of ultrametric spaces):

Theorem 6.2. *For every $d \geq 1$ the Urysohn ultrametric space of diameter d has finite big Ramsey degrees.*

Proof. We construct ultrametric space $\mathbf{U}_d$ as follows:

(1) Vertices of $\mathbf{U}_d$ are d-tuples of vertices of $\mathbf{P}$.
(2) The distance between vertices $(u_0, u_1, \dots u_{d-1})$ and $(v_0, v_1, \dots, v_{d-1})$ is the minimal ℓ such that for every $0 \leq i < d - \ell$ it holds that $u_i = v_i$.

Again, it is easy to verify that this is a universal ultrametric space. The finiteness of big Ramsey degrees now follows by an application of Theorem 1.1. □

Observe that by replacing $\mathbf{P}$ by ω above, the same result (with better bounds) follows by the infinite Ramsey theorem. Note that the construction above can be strengthened to the Λ-ultrametric spaces for a given finite lattice Λ [Bra17].

6.4. **Linear orders.** By fixing a linear extension of $\mathbf{P}$ one obtains an alternative proof of the Laver's result:

Corollary 6.3. *The order of rationals has finite big Ramsey degrees.*

While this may not be very powerful observation at its own, we will discuss its consequences in Corollary 6.5. Observe also that $\mathbf{P}$ has a natural linear extension in the form of the lexicographic order.

6.5. **Structures with unary relations.** Another particularly simple consequence of Theorem 1.1 is:

Corollary 6.4. *Let L be a finite language consisting of unary relational symbols. Then the universal homogeneous L-structure has finite big Ramsey degrees.*

Proof. For simplicity assume that L consists of single unary relation R. Then the universal L-structure can be represented using $\mathbf{P}$ as follows:

(1) Vertices are all pairs of distinct vertices of $\mathbf{P}$.
(2) Put vertex (u_0, u_1) to the relation R if and only if $u_0 \leq_{\mathbf{P}} u_1$.

□

6.6. **Free superpositions.** Recall that the *age* of a structure $\mathbf{M}$ is the set of all finite structures having an embedding to $\mathbf{M}$. Given a language L and its sub-language $L^- \subseteq L$, an L^--structure $\mathbf{M}$ is the *L^--reduct* of an L-structure $\mathbf{N}$ if $M = N$ and $R_{\mathbf{M}} = R_{\mathbf{N}}$ for every $R \in L^-$.

Let L and L' be languages such that $L \cap L' = \emptyset$. Let $\mathbf{M}$ be a homogeneous L-structure and $\mathbf{N}$ a homogeneous L'-structure. Then the *free superposition of* $\mathbf{M}$ *and* $\mathbf{N}$, denoted by $\mathbf{M} * \mathbf{N}$, is the homogeneous $L \cup L'$-structure whose age consists precisely of those finite $(L \cup L')$-structures with the property that their L-reduct is in the age of $\mathbf{M}$ and L'-reduct is in the age of $\mathbf{N}$ (see e.g. [Bod15]).

It follows from the product Ramsey argument that the free interposition of finitely many Ramsey classes with strong amalgamation property and no algebraicity is also Ramsey [Bod15, Lemma 3.22], see also [HN19, Proposition 4.45]. Similar general result is not known for big Ramsey structures. However, we can combine the above observation to the following corollary (of Theorem 4.8) which heads in this direction by providing means to interpose many of the known structures with finite big Ramsey degrees:

Corollary 6.5. *Let $\mathbf{M}$ be a homogeneous structure that is a free superposition of finitely many copies of structures from the following list (each in a language disjoint from the others):*

(1) the homogeneous universal partial order,
(2) the homogeneous universal triangle-free graph,
(3) the Urysohn S-metric space for a finite thin set S (for $S = \{0, 1, 2\}$ one obtains the Rado graph),
(4) the Urysohn ultrametric space of a finite diameter d,
(5) the order of rationals,
(6) the homogeneous universal structure in a finite unary relational language,

then $\mathbf{M}$ has finite big Ramsey degrees.

Proof. Let $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n$ be structures from the statement of the corollary, in mutually disjoint languages $L_1, L_2, \ldots, L_n$ such that for every $1 \leq i \leq n$ it holds that $\mathbf{M}_i$ is L_i-structure. Put $\mathbf{M} = \mathbf{M}_1 * \mathbf{M}_2 * \cdots * \mathbf{M}_n$.

As shown above, for each structure $\mathbf{M}_i$, $1 \leq i \leq n$, there exists a structure $\mathbf{N}_i$ and an embedding $e_i \colon \mathbf{M}_i \to \mathbf{N}_i$ such that $M_i = \binom{\mathbf{P}}{\mathbf{V}_i}$ for some finite structure $\mathbf{V}_i$ and $\mathbf{M}_i$ is represented using the partial order $\mathbf{P}$ (or its linear extension).

Now consider a $(L_1 \cup L_2 \cup \cdots \cup L_n)$-structure $\mathbf{N}$ defined as follows. The vertex set N of $\mathbf{N}$ consists of all n-tuples $(\vec{v}_1, \ldots, \vec{v}_n)$ with the property that for every $1 \leq i \leq n$ it holds that $\vec{v}_i$ is a vertex of $\mathbf{N}_i$. Denote by π_i the i-th projection $(\vec{v}_1, \ldots, \vec{v}_n) \mapsto \vec{v}_i$.

Now we define relations of $\mathbf{N}$. For every $1 \leq i \leq n$, consider structure $\mathbf{N}_i$:

(1) If $\mathbf{N}_i$ is a partial order, then the corresponding partial order of $\mathbf{N}$ is created by putting $u \leq v$ if and only if either $u = v$ or $\pi_i(u) \neq \pi_i(v)$ and $\pi_i(u) \leq_{\mathbf{N}_i} \pi_i(v)$.

(2) If $\mathbf{N}_i$ is homogeneous universal triangle free graph then we put u and v adjacent if and only if $\pi_i(u)$ is adjacent to $\pi_i(v)$ in $\mathbf{N}_i$.
(3) If $\mathbf{N}_i$ is the order of rationals, then the corresponding linear order of $\mathbf{N}$ is any linear order satisfying that π_i is a monotone function.
(4) If $\mathbf{N}_i$ is an S-metric space, then the corresponding metric space on $\mathbf{N}$ is created by defining a distance of u and v to be 0 if $u = v$, $\min(S \setminus \{0\})$ if $\pi_i(u) = \pi_i(v)$ and the distance of $\pi_i(u)$ and $\pi_i(v)$ otherwise.
(5) If $\mathbf{N}_i$ is an ultrametric space then the corresponding ultrametric space is created analogously, but by putting the distance to be 1 for every $u \neq v$, $\pi_i(u) = \pi_i(v)$.
(6) If $\mathbf{N}_i$ is a structure with unary relations, then for every relation $R \in L_i$ we put v to $R_{\mathbf{N}}$ if and only if $\pi_i(v) \in R$.

We say that substructure $\mathbf{A}$ of $\mathbf{N}$ is *transversal* if for every two vertices $(u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n) \in A$ and every $1 \leq i \leq n$ it holds that $u_i \neq v_i$. Observe that embeddings $e_i \colon \mathbf{M}_i \to \mathbf{N}_i$, $1 \leq i \leq n$, can be combined to an embedding $e \colon \mathbf{M} \to \mathbf{N}$ defined by putting $e(v) \mapsto (e_1(v), e_2(v), \ldots, e_n(v))$, and that the image $e(\mathbf{M})$ is transversal. One can also verify that for every $1 \leq i \leq n$ it holds that the age of $\mathbf{N}_i$ is the same as the age of the L_i-reduct of N. It follows that N and M have same ages. By universality of M it follows that there is also an embedding $f \colon \mathbf{N} \to \mathbf{M}$.

Fix a finite structure $\mathbf{A}$ and a finite coloring χ of $\binom{\mathbf{M}}{\mathbf{A}}$. Denote by $\mathcal{A}$ the set of all transversal structures in $\binom{\mathbf{N}}{\mathbf{A}}$. Consider a finite coloring χ' of $\mathcal{A}$ defined by $\chi'(\widetilde{\mathbf{A}}) = \chi(f^{-1}(\widetilde{\mathbf{A}}))$. For every $1 \leq i \leq n$ this coloring projects by π_i to a finite coloring of finite substructures of $\mathbf{N}_i$ and consequently also of $\mathbf{P}$. This follows from the fact that vertex set of $\mathbf{N}_j$ is $\binom{\mathbf{P}}{\mathbf{N}_j}$, for every $1 \leq j \leq n$ and thus preimages of vertices in projection π_i are all finite and isomorphic. By a repeated application of Theorem 4.8 it follows that $\mathbf{N}$ has finite big Ramsey degrees. By the existence of embedding e the corollary follows. □

Corollary 6.5 has further consequences. Superposing the Rado graph (which is the Urysohn S-metric space for $S = \{0, 1, 2\}$) and the universal homogeneous structure in language with one unary relation one can obtain that the random countable bipartite graph has finite big Ramsey degrees. This follows from the fact that the random countable bipartite graph can be defined in the superposition by considering only those edges where precisely one of the endpoints is in the unary relation.

Similarly, superposing the linear order with the universal homogeneous structure in language with one unary relation it follows that the homogeneous dense local order has big Ramsey degrees (as shown by Laflamme, Nguyen Van Thé and Sauer [LNVTS10]). Superposing multiple linear and partial orders leads to big Ramsey equivalents of results of Sokić [Sok13], Solecki and Zhao [SZ17], and Draganić and Mašulović [DM19].

7. Concluding remarks

7.1. **Bigger forbidden substructures and bigger arities.** The method presented in this paper can be used to strengthen Theorem 1.2 for free amalgamation classes in finite binary languages defined by finitely many forbidden irreducible substructures on at most 3 vertices.

For non-binary relations and bigger forbidden irreducible substructures it seems necessary to refine Theorem 2.1 for colouring multi-dimensional objects rather than words in a similar manner as in [BCH+19, BCH+20a]. This seems to further develop the link between constructions in structural Ramsey theory and the extension property for partial automorphisms [HKN19b].

7.2. **Optimality.** Big Ramsey degree a vertex in the universal homogeneous triangle-free graph was shown to be one by Komjáth and Rödl [KR86] in 1986. Big Ramsey degree of an edge is four as shown by Sauer [Sau98] in 1998. Proof of Theorems 1.1 and 1.2 can be refined to precisely describe the big Ramsey degrees in a similar way as was as done by Sauer [Sau06] for the random graph and Laflamme, Sauer and Vuksanovic for free binary structures [LSV06]. This leads to the big Ramsey structure as defined by Zucker [Zuc19]. This will appear in [BCH+20c].

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References

[BCH+19] Martin Balko, David Chodounský, Jan Hubička, Matěj Konečný, and Lluis Vena. Big Ramsey degrees of 3-uniform hypergraphs. *Acta Mathematica Universitatis Comenianae*, 88(3):415–422, 2019.

[BCH+20a] Martin Balko, David Chodounskỳ, Jan Hubička, Matěj Konečnỳ, and Lluis Vena. Big ramsey degrees of 3-uniform hypergraphs are finite. *arXiv:2008.00268*, 2020.

[BCH+20b] Martin Balko, David Chodounský, Jan Hubička, Matěj Konečný, and Lluis Vena. Big Ramsey degrees of unconstrained relational structures. (in preparation), 2020.

[BCH+20c] Martin Balko, David Chodounský, Jan Hubička, Matěj Konečný, Lluis Vena, and Andy Zucker. Big Ramsey degrees of triangle-free graphs. (in preparation), 2020.

[Bod15] Manuel Bodirsky. Ramsey classes: Examples and constructions. *Surveys in Combinatorics 2015*, 424:1, 2015.

[Bra17] Samuel Braunfeld. Ramsey expansions of Λ-ultrametric spaces. arXiv:1710.01193, 2017.

[Car87] Timothy J Carlson. An infinitary version of the Graham-Leeb-Rothschild theorem. *Journal of Combinatorial Theory, Series A*, 44(1):22–33, 1987.

[CDP20] Rebecca Coulson, Natasha Dobrinen, and Rehana Patel. Canonical partitions of relational structures. In final stages of preparation, 2020.

[CS84] Timothy J. Carlson and Stephen G. Simpson. A dual form of Ramsey's theorem. *Advances in Mathematics*, 53(3):265–290, 1984.

[Dev79] Denis Devlin. *Some partition theorems and ultrafilters on* ω. PhD thesis, Dartmouth College, 1979.

[DK16] Pandelis Dodos and Vassilis Kanellopoulos. *Ramsey theory for product spaces*, volume 212. American Mathematical Soc., 2016.

[DKT14] Pandelis Dodos, Vassilis Kanellopoulos, and Konstantinos Tyros. Measurable events indexed by words. *Journal of Combinatorial Theory, Series A*, 127:176–223, 2014.

[DLPS07] Christian Delhommé, Claude Laflamme, Maurice Pouzet, and Norbert W. Sauer. Divisibility of countable metric spaces. *European Journal of Combinatorics*, 28(6):1746–1769, 2007.

[DLPS08] Christian Delhommé, Claude Laflamme, Maurice Pouzet, and Norbert W. Sauer. Indivisible ultrametric spaces. *Topology and its Applications*, 155(14):1462–1478, 2008.

[DLS16] Natasha Dobrinen, Claude Laflamme, and Norbert Sauer. Rainbow Ramsey simple structures. *Discrete Mathematics*, 339(11):2848–2855, 2016.

[DM19] Nemanja Draganić and Dragan Mašulović. A Ramsey theorem for multiposets. *European Journal of Combinatorics*, 81:142–149, 2019.

[Dob19] Natasha Dobrinen. The Ramsey theory of Henson graphs. *arXiv:1901.06660, submitted*, 2019.

[Dob20a] Natasha Dobrinen. The Ramsey theory of the universal homogeneous triangle-free graph. *Journal of Mathematical Logic*, page 2050012, 2020.

[Dob20b] Natasha Dobrinen. Ramsey theory on infinite structures and the method of strong coding trees. In Adrian Rezus, editor, *Contemporary Logic and Computing*, Landscapes in Logic. College Publications, 2020.

[DR12] Domingos Dellamonica and Vojtěch Rödl. Distance preserving Ramsey graphs. *Combinatorics, Probability and Computing*, 21(04):554–581, 2012.

[EH74] Paul Erdős and András Hajnal. Unsolved and solved problems in set theory. In *Proceedings of the Tarski Symposium (Berkeley, Calif., 1971), Amer. Math. Soc., Providence*, volume 1, pages 269–287, 1974.

[FK89] Hilel Fürstenberg and Yitzhak Katznelson. Idempotents in compact semigroups and Ramsey theory. *Israel Journal of Mathematics*, 68(3):257–270, 1989.

[Fou97] Willem L Fouché. Symmetry and the Ramsey degree of posets. *Discrete Mathematics*, 167:309–315, 1997.

[GR71] Ronald L. Graham and Bruce L. Rothschild. Ramsey's theorem for n-parameter sets. *Transactions of the American Mathematical Society*, 159:257–292, 1971.

[Hed69] Zeněk Hedrlín. On universal partly ordered sets and classes. *Journal of Algebra*, 11(4):503–509, 1969.

[HKN19a] Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil. Ramsey expansions and EPPA for s-metric spaces. in preparation, 2019.

[HKN19b] Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil. All those EPPA classes (strengthenings of the Herwig-Lascar theorem). arXiv:1902.03855, 2019.

[HL66] James D. Halpern and Hans Läuchli. A partition theorem. *Transactions of the American Mathematical Society*, 124(2):360–367, 1966.

[HN05a] Jan Hubička and Jaroslav Nešetřil. Finite presentation of homogeneous graphs, posets and Ramsey classes. *Israel Journal of Mathematics*, 149(1):21–44, 2005.

[HN05b] Jan Hubička and Jaroslav Nešetřil. Universal partial order represented by means of oriented trees and other simple graphs. *European Journal of Combinatorics*, 26(5):765–778, 2005.

[HN11] Jan Hubicka and Jaroslav Nešetril. Some examples of universal and generic partial orders. *Model Theoretic Methods in Finite Combinatorics*, pages 293–318, 2011.

[HN19] Jan Hubička and Jaroslav Nešetřil. All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). *Advances in Mathematics*, 356C:106791, 2019.

[Kar13] Nikolaos Karagiannis. A combinatorial proof of an infinite version of the Hales–Jewett theorem. *Journal of Combinatorics*, 4(2):273–291, 2013.

[KPT05] Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorčević. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.

[KR86] Péter Komjáth and Vojtech Rödl. Coloring of universal graphs. *Graphs and Combinatorics*, 2(1):55–60, 1986.

[Lar08] Jean A Larson. Counting canonical partitions in the random graph. *Combinatorica*, 28(6):659–678, 2008.

[Lav84] Richard Laver. Products of infinitely many perfect trees. *Journal of the London Mathematical Society*, 2(3):385–396, 1984.

[LNVTS10] Claude Laflamme, Lionel Nguyen Van Thé, and Norbert W. Sauer. Partition properties of the dense local order and a colored version of Milliken's theorem. *Combinatorica*, 30(1):83–104, 2010.

[LSV06] Claude Laflamme, Norbert W Sauer, and Vojkan Vuksanovic. Canonical partitions of universal structures. *Combinatorica*, 26(2):183–205, 2006.

[Mac11] Dugald Macpherson. A survey of homogeneous structures. *Discrete Mathematics*, 311(15):1599–1634, 2011. Infinite Graphs: Introductions, Connections, Surveys.

[Maš18] Dragan Mašulović. Pre-adjunctions and the Ramsey property. *European Journal of Combinatorics*, 70:268–283, 2018.

[Mil79] Keith R. Milliken. A Ramsey theorem for trees. *Journal of Combinatorial Theory, Series A*, 26(3):215–237, 1979.

[Neš07] Jaroslav Nešetřil. Metric spaces are Ramsey. *European Journal of Combinatorics*, 28(1):457–468, 2007.

[NR75a] Jaroslav Nešetril and Vojtěch Rödl. A Ramsey graph without triangles exists for any graph without triangles. *Infinite and finite sets III*, 10:1127–1132, 1975.

[NR75b] Jaroslav Nešetřil and Vojtěch Rödl. Type theory of partition properties of graphs. In *Recent advances in graph theory*, pages 405–412. Academia Prague, 1975.

[NR77] Jaroslav Nešetřil and Vojtěch Rödl. A structural generalization of the Ramsey theorem. *Bulletin of the American Mathematical Society*, 83(1):127–128, 1977.

[NR84] Jaroslav Nešetřil and Vojtěch Rödl. Combinatorial partitions of finite posets and lattices—Ramsey lattices. *Algebra Universalis*, 19(1):106–119, 1984.

[NR89] Jaroslav Nešetřil and Vojtěch Rödl. The partite construction and Ramsey set systems. *Discrete Mathematics*, 75(1):327–334, 1989.

[NR18] Jaroslav Nešetřil and Vojtěch Rödl. Ramsey partial orders from acyclic graphs. *Order*, 35(2):293–300, 2018.

[NVT09] Lionel Nguyen Van Thé. Ramsey degrees of finite ultrametric spaces, ultrametric Urysohn spaces and dynamics of their isometry groups. *European Journal of Combinatorics*, 30(4):934–945, 2009.

[NVT10] Lionel Nguyen Van Thé. *Structural Ramsey Theory of Metric Spaces and Topological Dynamics of Isometry Groups*. Memoirs of the American Mathematical Society. American Mathematical Society, 2010.

[Prö13] Hans Jürgen Prömel. *Ramsey Theory for Discrete Structures*. Springer International Publishing, 2013.

[PT80] Aleš Pultr and Věra Trnková. *Combinatorial, Algebraic, and Topological Representations of Groups, Semigroups, and Categories*. Mathematical Studies. North-Holland Publishing Company, 1980.

[PTW85] Madeleine Paoli, William T. Trotter, and James W. Walker. Graphs and orders in Ramsey theory and in dimension theory. In Ivan Rival, editor, *Graphs and Order*, volume 147 of *NATO AST series*, pages 351–394. Springer, 1985.

[PV85] Hans Jürgen Prömel and Bernd Voigt. Baire sets of k-parameter words are Ramsey. *Transactions of the American Mathematical Society*, 291(1):189–201, 1985.

[Sau98] Norbert W. Sauer. Edge partitions of the countable triangle free homogeneous graph. *Discrete Mathematics*, 185(1-3):137–181, 1998.

[Sau06] Norbert W. Sauer. Coloring subgraphs of the Rado graph. *Combinatorica*, 26(2):231–253, 2006.

[Sau12] Norbert W. Sauer. Vertex partitions of metric spaces with finite distance sets. *Discrete Mathematics*, 312(1):119–128, 2012.

[Sau13] Norbert W. Sauer. Distance sets of Urysohn metric spaces. *Canadian Journal of Mathematics*, 65(1):222–240, 2013.

[Sau20] Norbert W. Sauer. Colouring homogeneous structures. *arXiv:2008.02375*, 2020.

[Sok13] Miodrag Sokić. Ramsey property, ultrametric spaces, finite posets, and universal minimal flows. *Israel Journal of Mathematics*, 194(2):609–640, 2013.

[SZ17] Sławomir Solecki and Min Zhao. A Ramsey theorem for partial orders with linear extensions. *European Journal of Combinatorics*, 60:21–30, 2017.

[Tod10] Stevo Todorcevic. *Introduction to Ramsey spaces*, volume 174. Princeton University Press, 2010.

[Zuc19] Andy Zucker. Big Ramsey degrees and topological dynamics. *Groups, Geometry, and Dynamics*, 13(1):235–276, 2019.

[Zuc20] Andy Zucker. A note on big Ramsey degrees. *arXiv:2004.13162*, 2020.

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