Minimal Infinite Topological Spaces
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defined in (3). Let $\mathbf{r} \in S^1$. Recall that $c^*(\mathbf{r}) = R_{\mathbf{r}}(\mathbf{v})$. By direct computation,

$$\mathbf{v} \cdot T(\mathbf{r}) = f_2(\mathbf{r}) \cdot T(\mathbf{r}).$$

Hence $f_2(\mathbf{r}) = \mathbf{v}$ or $f_2(\mathbf{r}) = R_{\mathbf{r}}(\mathbf{v})$. But the former case occurs only when $\mathbf{r} = (0, \pm 1)$, and then $R_{\mathbf{r}}(\mathbf{v}) = \mathbf{v} = f_2(\mathbf{r})$.

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MINIMAL INFINITE TOPOLOGICAL SPACES

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In this note we will describe a collection of five infinite topological spaces having the property that every infinite space contains one of the members of the collection as a subspace. We denote the set of natural numbers by ω. Consider the following five topologies with underlying set ω:

(i) *discrete*: all subsets of ω are open;
(ii) *indiscrete*: the only open sets are ω and $\varnothing$;
(iii) *cofinite*: the open sets are ω, $\varnothing$, and all subsets of ω whose complements are finite;
(iv) *initial segment*: the open sets are ω, $\varnothing$, and all sets of the form $[0,n] = \{k \in \omega : k \leqslant n\}$ where $n \in \omega$;
(v) *final segment*: the open sets are ω, $\varnothing$, and all sets of the form $[n,\omega] = \{k \in \omega : n \leqslant k\}$ where $n \in \omega$.

We will establish the following result.

THEOREM. *Every infinite topological space contains one of the preceding five spaces as a subspace.*

Note that no two of the five spaces are homeomorphic, and each of the five spaces is homeomorphic to all of its infinite subspaces. It follows that these five spaces form the smallest collection of infinite spaces satisfying the conclusion of the theorem.

Before proceeding with the proof, let us compare our result with the analogous situation in some other mathematical structures.

(a) Let G be a graph with infinitely many vertices. Ramsey ([3], cf. [2, page 15]) showed that

G must contain as a subgraph either the complete graph on countably many vertices or the totally disconnected graph on countably many vertices.

(b) Let P be a partially ordered set with infinitely many elements. From Ramsey's theorem it follows that P must contain either an infinite chain or an infinite antichain. Since any infinite chain will contain ω if it is well ordered and ω^* (the dual of ω) if it is not, P must contain one of the partially ordered sets of Figure 1 as a subset.

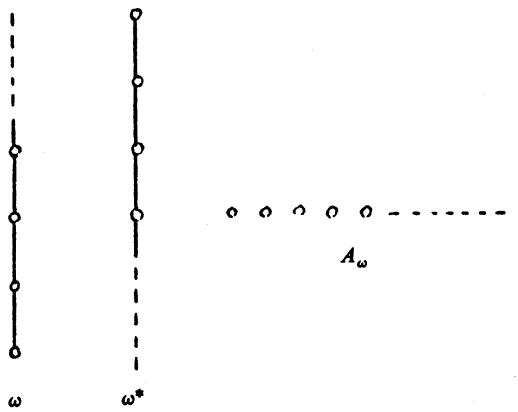

FIG. 1.

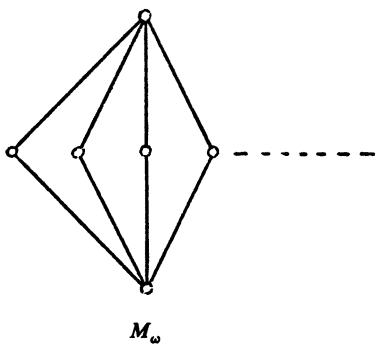

FIG. 2.

(c) Let L be a lattice with infinitely many elements. T. P. Whaley [4, Corollary 2.4] showed that L must contain a sublattice isomorphic to ω, ω^*, or the lattice M_ω of Figure 2. We feel it worthwhile to include a proof of this result. If L contains an infinite chain, then L contains ω or ω^*, as desired; therefore we assume that all chains in L are finite. (This is a very strong assumption. For example, it implies that L is complete.) Recall that for $a, b \in L$, a is said to be a *lower cover* of b if $a < b$ and no element of L lies strictly between a and b. Every non-empty subset of L contains a maximal and a minimal element; thus every element of L other than the least element 0 contains a lower cover, and we may choose $a \in L$ such that the interval $[0, a]$ is infinite but $[0, c]$ is finite for all lower covers c of a. The set C of all lower covers of a must then be infinite. Let $c \in C$. Since $\{x \wedge c : x \in C\} \subseteq [0, c]$ is finite, there are an element b of L and an infinite subset C_1 of C such that $c \wedge x = b$ for all x in C_1. Choose an element d of L which is maximal with respect to the property that there exists $c \in C$ and an infinite subset C_1 of C such that $c \wedge x = d$ for all $x \in C_1$. The preceding argument shows that elements having this property exist, hence such a choice is possible. Choose $c_0 \in C$ and an infinite subset C_1 of C such that $c_0 \wedge x = d$ for all $x \in C_1$. Now choose $c_1 \in C_1$. Since $[0, c_1]$ is finite, there is an infinite subset C_2 of C_1 such that $c_1 \wedge x = c_1 \wedge y$ for all x and y in C_2. Furthermore, by the choice of d, we must have $c_1 \wedge x = d$ for all x in C_2. Choosing $c_2 \in C_2$ and continuing this procedure, we obtain a sublattice $\{a, d, c_0, c_1, c_2, \ldots\}$ isomorphic to M_ω.

(d) In striking contrast is the situation for groups. Here, the problem of finding all "minimal" infinite groups seems to be very difficult. In fact, this problem may not even be meaningful! For example, there might exist non-isomorphic infinite groups G and H such that G contains a subgroup isomorphic to H, H contains a subgroup isomorphic to G, and every infinite subgroup of G or H is isomorphic to G or H; in this case no infinite subgroup of G or H would seem to qualify as a member of the list of "minimal" infinite groups. If such a list exists, it must be infinite (for each prime p, consider the group of rational numbers between 0 and 1 with denominators a power of p, under addition modulo 1). Furthermore, it must contain at least one nonabelian group, since infinite Burnside groups have the property that every abelian subgroup is finite (cf. [1, page 34]).

Proof of the theorem. Let X be an infinite topological space. For a subset S of X we denote the closure of S in X by $\mathrm{cl}S$. Consider the equivalence relation $\sim$ on X defined by $a \sim b$ iff $\mathrm{cl}\{a\} = \mathrm{cl}\{b\}$. The subspace topology on each equivalence class is the indiscrete topology. Therefore if some equivalence class modulo $\sim$ is infinite, X contains a subspace homeomorphic to (ii) above. So assume each equivalence class is finite. In particular, there are infinitely many distinct equivalence classes. Choosing one element from each class gives rise to an infinite subspace of X which satisfies the T_0 separation axiom. We may therefore assume, without loss of generality, that X itself is T_0.

Now consider the relation $\leqslant$ on X defined by $a \leqslant b$ iff $a \in \mathrm{cl}\{b\}$. Notice that since X is T_0, $\leqslant$ is antisymmetric, and it follows that $\leqslant$ is a partial order on X. If $x < y$, then every open subset of X which contains x also contains y; furthermore, $y \notin \mathrm{cl}\{x\}$, so there is an open subset of X containing y but not x. The partially ordered set $(X, \leqslant)$ is infinite, and hence contains a subset isomorphic to one of the three partially ordered sets of Figure 1. Let $A = \{a_1, a_2, \dots\}$ with $a_1 < a_2 < \cdots$ be a subset of X isomorphic to ω. Let G be a nonempty open subset of A, and set $m = \min\{k \in \omega | a_k \in G\}$. Then $G\{a_k | k \geqslant m\}$; moreover, for each n there is an open subset of X containing a_n but not a_{n-1}, whence $\{a_k | k \geqslant n\}$ is open in A for each n. Thus, A with the subspace topology is homeomorphic to ω with the final segment topology (v). Similarly, let $B = \{b_1, b_2, \dots\}, b_1 > b_2 > \cdots$, be a subset of X isomorphic to ω^*, and let $G \neq B$ be a nonempty open subset of B. It follows that $m' = \max\{k \in \omega | b_k \in G\}$ exists and that $G = \{b_k | k \leqslant m'\}$; moreover, for each n there is an open subset of X containing b_n but not b_{n+1}, whence $\{b_k | k \leqslant n\}$ is open in B for each n. Thus, B with the subspace topology is homeomorphic to ω with the initial segment topology (iv). Therefore we assume $(X, \leqslant)$ contains an infinite antichain, and it follows that the subspace topology on this antichain will be T_1. So, without loss of generality, we assume X is T_1.

Thus points are closed, and so every cofinite subset of X is open in X. If the topology on X is the cofinite topology, then any countable subspace of X is homeomorphic to (iii) above. Thus we may assume that no infinite subspace of X carries the cofinite topology. In particular, there is a non-empty open set U_0 in X whose complement is infinite. Choose $x_0 \in U_0$. Since the subspace topology on $X - U_0$ is not the cofinite topology, there is an open set U_1' in X such that $U_1' - U_0$ is non-empty and such that $X - (U_0 \cup U_1')$ is infinite. Choose $x_1 \in U_1' - U_0$ and let $U_1 = U_1' - \{x_0\}$. Then U_0, U_1 are open in X, $x_0 \in U_0 - U_1$, $x_1 \in U_1 - U_0$, and $X - (U_0 \cup U_1)$ is infinite. Suppose we have chosen open sets $U_0, U_1, \dots, U_n$ in X and points $x_0, x_1, \dots, x_n$ of X such that

(a) for all $i, j \leqslant n$, $x_i \in U_j$ iff $i = j$, and

(b) $X - \cup_{i=1}^{n} U_i$ is infinite.

Then $X - \cup_{i=1}^{n} U_i$ does not carry the cofinite topology, so there is an open set U_{n+1}' in X such that $(X - \cup_{i=1}^{n} U_i) \cap U_{n+1}' \neq \varnothing$ and $(X - \cup_{i=1}^{n} U_i) - U_{n+1}'$ is infinite. Choose a point x_{n+1} in $U_{n+1}' - \cup_{i=1}^{n} U_i$ and let $U_{n+1} = U_{n+1}' - \{x_0, x_1, \dots, x_n\}$. Then $U_0, U_1, \dots, U_{n+1}$ and $x_0, x_1, \dots, x_{n+1}$ satisfy (a) and (b) above for $n+1$. By induction, we obtain a sequence of open sets $\{U_0, U_1, U_2, \dots\}$ in X and points $\{x_0, x_1, x_2, \dots\}$ in X satisfying (a) and (b) for all n. Let $Y = \{x_0, x_1, x_2, \dots\}$. By (a), points of Y are open in Y. Thus Y is discrete and homeomorphic to (i) above. ■

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