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A GAME-THEORETIC PROOF OF ANALYTIC RAMSEY THEOREM

by KAZUYUKI TANAKA in Sendai (Japan)

Abstract

We give a simple game-theoretic proof of Silver's theorem that every analytic set is Ramsey. A set $P$ of subsets of $\omega$ is called Ramsey if there exists an infinite set $H$ such that either all infinite subsets of $H$ are in $P$ or all out of $P$. Our proof clarifies a strong connection between the Ramsey property of partitions and the determinacy of infinite games.

MSC: 03E15, 03E60, 05A17.

Key words: Analytic Ramsey theorem, determinacy of infinite games.

Let $\omega$ be the set of non-negative integers. For an infinite subset $X$ of $\omega$ let $[X]^n$ denote the set of all subsets of $X$ with exactly $n$ elements ($n \in \omega$). Suppose that $[X]^n$ is partitioned into $P_1$ and $P_2$. Then the classical version of RAMSEY's theorem asserts that there is an infinite subset $H$ of $\omega$ such that either $[H]^n \subseteq P_1$ or $[H]^n \subseteq P_2$.

In this paper, we discuss the following natural generalization of RAMSEY's theorem. For any infinite subset $X$ of $\omega$, let $[X]^n$ denote the set of all infinite subsets of $X$. We say that a partition $P \subseteq [\omega]^n$ is Ramsey if there exists an infinite subset $H$ of $\omega$ such that either $[H]^n \subseteq P$ or $[H]^n \subseteq [\omega]^n - P$. By the axiom of choice, we can easily construct a partition which is not Ramsey. But it is natural to ask which sets, in terms of logical complexity, are Ramsey. This problem was first posed by DANA SCOTT (unpublished) in the mid-1960's.

After GALVIN-PRIKRY [3] proved that all Borel sets are Ramsey, SILVER [7] has given a complete answer to the problem, by showing in ZF + DC that (i) $\Sigma_1^1$ sets are Ramsey, (ii) the statement of $\Delta_1^1$ Ramseyness contradicts with GÖDEL's axiom $V = L$, and (iii) if there is a measurable cardinal, then $\Sigma_1^1$ sets are Ramsey. Recall some notation. A set is $\Sigma_1^1$ (or analytic) iff it is a projection of a Borel (\Delta_1^1) set iff it is defined by a $\Sigma_1^1$ formula with parameters. A set is $\Sigma_1^1$ iff it is a projection of a co-analytic (\Pi_1^1) set iff it is defined by a $\Sigma_1^1$ formula with parameters. A set is $\Delta_1^1$ iff it can be defined both by a $\Sigma_1^1$ formula and a $\Pi_1^1$ formula without parameter. For more details, see MOSCHOVAKIS [6]. A nice exposition of SILVER's theorem can be found in BOLLOBÁS [2].

In [4], KASTANAS shows that the Ramsey property of a partition $P$ can be deduced from the determinacy of a certain infinite game of the same logical complexity as $P$. Although his proof has many interesting points, it does not provide an alternative proof of SILVER's theorem since analytic determinacy over the reals is not provable in ZF + DC. So we improve his game construction by an unfolding trick. Our game is similar to an asymmetric game of KECHRIS [5] which KASTANAS might try to use at the end of the paper, but it is actually more elementary. We will prove analytic Ramseyess from $\Sigma_2^1$ determinacy over the reals, and generally $\Sigma_{n+1}^1$ Ramseyess from $\Sigma_n^1$ determinacy over the reals. Since $\Sigma_0^1$ determinacy over the reals is provable in ZF, this gives another proof of SILVER's theorem.
We will treat only the lightface statements, since the boldface versions (including parameters) are straightforward from them by the usual relativization argument. Let $P$ be a $\Sigma^1_2$ subset of $[\omega]^{\omega}$. Then there exists a $\Sigma^1_2$ formula $\varphi(f, X)$ such that $P(X) \iff \exists f \in 2^\omega \varphi(f, X)$. We define the game $G_P$ as follows:

\[
\begin{align*}
&\text{I} \\
&d_0, A_0 \\
&d_1, A_1 \\
&\vdots \\
&n_0, B_0 \\
&n_1, B_1 \\
&\vdots 
\end{align*}
\]

The rules of $G_P$ are

(i) $A_0 \in [\omega]^{\omega}$, $A_{i+1} \in [B_i]^{\omega}$, and $d_i = 0$ or 1,

(ii) $B_i \in [A_i]^{\omega}$, $n_i \in A_i$, and $n_i < b$ for all $b \in B_i$.

The first person who disobeys one of the above rules loses the game. When all the rules are obeyed, player I wins iff $\varphi(f, H)$ holds, where $f(i) = d_i$ for all $i \in \omega$ and $H = \{n_0, n_1, \ldots\}$. Thus $G_P$ is a $\Sigma^1_2$ game and I is a $\Pi^0_2$ player.

For a $\Sigma^1_{n+1}$ partition $P$, we also define the game $G_P$ in the same way. Supposing $P(X) \iff \exists f \in 2^\omega \varphi(f, X)$ with $\varphi \in \Pi^1_{n+1}$, player I wins iff $\varphi(f, H)$ holds, and so $G_P$ is a $\Sigma^1_{n+1}$ game and I is a $\Pi^1_n$ player.

Regarding a $\Sigma^1_n$ partition $P (n \geq 1)$ and its associated game $G_P$, we have

Theorem. (a) Every analytic set is Ramsey. (b) $\Sigma^1_n$-determinacy over the reals implies $\Sigma^1_{n+1}$-Ramsey. In particular, if there is a measurable cardinal ($\geq 2^\omega$), every $\Sigma^1_1$ set is Ramsey.

Note that WOLFE’s proof of $\Sigma^1_2$-determinacy and MARTIN’s proof of analytic determinacy (based on a measurable cardinal) both can be carried out for the games over the reals as well as the natural numbers (see MOSCHOVAKIS [6]). Now we prove the theorem.

Proof. (a) Let $\sigma$ be a winning strategy for I. We will construct an infinite set $H$ such that for each $X = \{x_0, x_1, \ldots\} \in [H]^{\omega}$, there is a play $(d_0, A_0)^\omega \cdots (x, B_i)^\omega \cdots$ which is consistent with $\sigma$, i.e.,

$$(d_i, A_i) = \sigma((d_0, A_0)^\omega \cdots (x, B_0)^\omega \cdots (x, B_{i-1})^\omega) \quad \text{for all } i \in \omega.$$ 

Since I wins at this play, we have $\varphi(f, X)$, where $f(i) = d_i$ for all $i \in \omega$, and so $P(X)$ holds.

If a partial play $(d_0, A_0)^\omega \cdots (n, B_{i-1})^\omega \cdots$ realizes a sequence $(n_0, n_1, \ldots, n_{i-1})$. To construct a set $H = \{n_0, n_1, \ldots\}$ of the above property, we simultaneouly build an $\omega$-sequence of finite trees $T_0 \subseteq T_1 \subseteq \ldots$ such that for each $i \in \omega$, $T_i$ consists of certain $\varphi$-consistent partial plays extending plays in $T_{i-1}$ and every subset of $n_0, n_1, \ldots, n_{i-1}$ is realized in a partial play in $T_i$. Once such $T_i$’s are built, it is clear that for each $X \in [H]^{\omega}$ there is a path through $\cup T_i$ generates (realizes) $X$. We now show the inductive construction of $T_i$’s first move given by $\sigma$. Put $T_0$.

In the induction step, we assume $T_i$ constructed, and additionally assume in $T_i$ end with $(d, A)$ such that $\varphi(f, X)$ holds and $n_i = \min \{n \in A : \exists d \in A \varphi(f, X)\}$. Let $n_i$ be the least such $n_i$.

$Y_0 = X \cap \{n_0\}, \quad (d, A)$,

Then we define $T_{i+1}$ as follows:

$$(d, A) \in T_i \cup \{p_\varphi(n_i, Y_0)\}$$

It is obvious that any subset of $\omega$ is Ramsey, and that all the partial plays in $T_i$’s are of (a).

(b) The basic idea of the following argument is as follows. We need extra treatment for the case by I’s winning strategy in part (b) of $\Sigma^1_{n+1}$ infinite set $H$ such that for each $n \in H$ it generates $X$ and $f$. Clearly such a $H$ exists.

We here say that a $\varphi$-consistent player $\sigma$ realizes the pair of sequences $(H, f) = \{n_0, n_1, \ldots\}$ with $f(i) = d_i$ for all $i \in \omega$, $T_i$ consists of some $\varphi$-consistent partial plays such as $n_i$ are the first number in each subset $s$ of $n_0, n_1, \ldots, n_{i-1}$: there is a partial play in $T_i$ which realizes each $X \in [H]^{\omega}$ and for each $f$ of (b) a partial pair $(X, f)$.

Before the construction of such sets $H$ and $f$.

Lemma. (cf. KASTANAS’ proof) Let $\sigma$ be a winning strategy over the reals $\in [\omega]^{\omega}$ with the last move $(n, B)$ such that $\varphi(f, X)$ holds and $d_i = (d_i = 0$ or 1) there exist $X$ and $f$

Proof of the Lemma. We define

$$(m_0, Y_0) = \tau(p_\varphi(0, B))$$

Then put $Y_0 = \{m_0, m_1, \ldots\}$. No $$(m_0, Y_0) = \tau(p_\varphi(1, B))$$

Then put $Y_0 = \{m_0, m_1, \ldots\}$. Clearly $$(m_0, Y_0) = \tau(p_\varphi(1, B))$$

We are now back to the construction argument, where $\sigma$ realizes the pair of the empty set $\{n_0, n_1, \ldots, n_{i-1}\}$ and $T_i \subseteq T_{i+1}$ and $T_{i+1}$ there is an infinite set $\tau$ such that.

Let $\{p_0, p_1, \ldots, p_{k-1}\}$ be an enumeration of the partial plays in $T_{i+1}$ and $T_{i+1}$.
We now show the inductive construction of \( H = \{ n_0, n_1, \ldots \} \) and \( T_i \)'s. Let \((d_0, A_0)\) be player I's first move given by \( \sigma \). Put \( T_0 = \{ (d_0, A_0) \} \). The empty sequence is realized by \((d_0, A_0)\). For the induction step, we assume that \( \{ n_0, n_1, \ldots , n_{i-1} \} \) and \( T_0 \subseteq T_1 \subseteq \ldots \subseteq T_i \) have been constructed, and additionally assume that there is an infinite set \( X_i \) such that all the partial plays in \( T_i \) end with \((d, A)\) such that \( X_i \subseteq A \). Let \( \{ p_0, p_1, \ldots , p_{k-1} \} \) be an enumeration of the elements of \( T_i \). Let \( n_i \) be the least element of \( X_i \). We define \( d_j \) \((j < k)\) and \( Y_j \) \((j \leq k)\) by

\[
Y_0 = X_i \setminus \{ n_i \}, \quad (d_j, Y_{j+1}) = \sigma(p_j^n(n_i, Y_j)).
\]

Then we define \( T_{i+1} \) as follows:

\[
T_{i+1} = T_i \cup \{ p_j^n(n_i, Y_j)(d_j, Y_{j+1}) : j < k \}.
\]

It is obvious that any subset of \( \{ n_0, n_1, \ldots , n_i \} \) is realized by a play in \( T_{i+1} \). We also notice that all the partial plays in \( T_{i+1} \) end with supersets of \( X_{i+1} = Y_i \). This completes the proof of (a).

(b) The basic idea of the following proof is the same as that of part (a). However, we here need some extra treatment for the sequence \( f = \{ d_0, d_1, \ldots \} \), which was automatically decided by I's winning strategy in part (a). Let \( \tau \) be a winning strategy for II. We will construct an infinite set \( H \) such that for each \( X \in [H]^\omega \) and for each \( f \in 2^\omega \) there is a \( \tau \)-consistent play which generates \( X \) and \( f \). Clearly such an \( H \) is homogeneous for \( \neg P(X) \).

We here say that a \( \tau \)-consistent partial play \((d_0, A_0)^\wedge(n_0, B_0)^\wedge \ldots (d_{i-1}, A_{i-1})^\wedge(n_{i-1}, B_{i-1})^\wedge\) realizes the pair of sequences \((n_0, n_1, \ldots , n_{i-1}) \) and \((d_0, d_1, \ldots , d_{i-1}) \). We construct \( H = \{ n_0, n_1, \ldots \} \) together with an \( \omega \)-sequence of finite trees \( T_0 \subseteq T_1 \subseteq \ldots \) such that for each \( i \in \omega \), \( T_i \) consists of some \( \tau \)-consistent partial plays extending plays in \( T_{i-1} \), and such that for each subset \( s \) of \( n_0, n_1, \ldots , n_{i-1} \) and for each sequence \( d \) of 0's and 1's with the same length as \( s \) there is a partial play in \( T_i \) which realizes the pair \((s, d)\). If we have such \( H \) and \( T_i \)'s, then for each \( X \in [H]^\omega \) and for each \( f \in 2^\omega \) there is a path through \( \bigcup T_i \) generating (realizing) the pair \((X, f)\).

Before the construction of such \( H \) and \( T_i \)'s, we prove the following lemma:

**Lemma.** (cf. Kastanas' \( \sigma_0 \) Lemma [4]). Let \( C \) be an infinite subset of \( \omega \). For every partial play \( p \) with the last move \((n, B)\) such that \( B \supseteq C \) there exists a set \( A \supseteq C \) such that for every \( m \in A \) and \( d \) (= 0 or 1) there exist \( X \) and \( Y \) such that \( \tau(p^\wedge(d, X)) = (m, Y) \) and \( Y \supseteq A - \{ m \} \).

**Proof of the Lemma.** We first define the sequence of pairs \( (m_i, Y_i) \) as follows:

\[
(m_0, Y_0) = \tau(p^\wedge(0, B)), \quad (m_{i+1}, Y_{i+1}) = \tau(p^\wedge(0, Y_i)), \quad \text{for } i \in \omega.
\]

Then put \( Y_\omega = \{ m_0, m_1, \ldots \} \). Next define the sequence of pairs \( (m'_i, Y'_i) \) as follows:

\[
(m'_0, Y'_0) = \tau(p^\wedge(1, Y_\omega)), \quad (m'_{i+1}, Y'_{i+1}) = \tau(p^\wedge(1, Y'_i)), \quad \text{for } i \in \omega.
\]

Then put \( Y'_\omega = \{ m'_0, m'_1, \ldots \} \). Clearly, \( A = Y'_\omega \) satisfies the lemma.

We are now back to the construction of \( H \) and \( T_i \)'s. Let \( T_0 = \{ \emptyset \} \). The empty sequence \( \emptyset \) realizes the pair of the empty sequences \( (\emptyset, \emptyset) \). For the induction step, we assume that \( \{ n_0, n_1, \ldots , n_{i-1} \} \) and \( T_0 \subseteq T_1 \subseteq \ldots \subseteq T_i \) have been constructed, and additionally assume that there is an infinite set \( C_i \) such that all the partial plays in \( T_i \) end with \((n, Y)\) such that \( C_i \subseteq Y \). Let \( \{ p_0, p_1, \ldots , p_{k-1} \} \) be an enumeration of the elements of \( T_i \).
We then apply the above lemma repeatedly as follows: let $A_0$ be a set obtained from $C_i$ and $p_0$ in the lemma, and $A_1$ a set obtained from $A_0$ and $p_1$ in the lemma, ... , and $A_{k-1}$ a set obtained from $A_{k-2}$ and $p_{k-1}$ in the lemma. Then let $n_i$ be the least element of $A_{k-1}$, and $C_{i+1} = A_{i+1} - \{n_i\}$. By the lemma, there exist $X_j$, $Y_j$, $X'_j$, and $Y'_j$ ($j < k$) such that all of them are supersets of $C_{i+1}$ and
\[
\tau(p_i \circ (0, X_j)) = (n_i, Y_j), \quad \tau(p_i \circ (1, X'_j)) = (n_i, Y'_j), \text{ for } j < k.
\]

Finally, we define $T_{i+1}$ as follows:
\[
T_{i+1} = T_i \cup \{p_i \circ (0, X_j) \circ (n_i, Y_j) : j < k\} \cup \{p_i \circ (1, X'_j) \circ (n_i, Y'_j) : j < k\}
\]

Obviously $T_{i+1}$ satisfies all the required conditions. This completes the proof.

Recently, Blass [1] also stresses a connection between partitions and games, though he does not establish such an effective relation as in this paper.

References


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§ 0. Introduction

The principal purpose of this paper is to establish connections between functions $+$, $\cdot$ and a constant $0$ and then prove a theorem of systems of inequalities.

MSC: 03F30, 03B25.

Key words: recursive arithmetic system of equations.

An elementary system is an axiomatic system with first-order rules about equations over the natural numbers. In this paper, we shall define a system of inequalities over the natural numbers to be semi-complete if any true inequality in the system is complete. Therefore, if an elementary system is complete, it contains a decision procedure for determining whether or not a given inequality is true. However, elementary systems do not have decision procedures for determining whether or not a given inequality is true.

In section 1, we first develop the formal system AS and list some of the theorems and rules will be presented in section 1. Then, in section 2, we will prove theorems about system AS.

In this paper, $\alpha = (\langle a \rangle \beta)$ is read as $\alpha$ or replacement, and $\alpha = (\langle a \rangle \beta)$ is obtained by applying rule (a). We refer readers to [4], [5]. But because of the limitations of the printing process, we do not include the details of the proof.

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AN ELEMENTARY SYSTEM AND ITS SEMI-COMPLETENESS

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Abstract

The author establishes an elementary system of inequalities containing functions $+$, $\cdot$ and a constant $0$ and then proves a theorem of systems of inequalities.

MSC: 03F30, 03B25.

Key words: recursive arithmetic system of equations.