BORELL SETS ARE RAMSEY

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1 INTRODUCTION

This paper is my rewrite of the first part of the paper which was published in the Journal of Symbold Logic in 1973 by Fred Galvin and Karel Prikry. While doing that I've tried to achieve the following goals:

- To go over their proof and fill the gaps in order to make their proof easy to understand to anyone who knows what the terms "open set" and "continuous function" mean.
- To show the connection between colorings and the defintions used in their text.
- To generalize their proof of 2 coloring, to any finite coloring given by some Borel function.

The proof contains 2 main steps. We first show that all open sets are Ramsey. After that we show that Ramsey sets form σ-algebra. After that we use the fact that Borel algebra is the smallest σ-algebra that contains all the open sets and conclude that Borel sets are Ramsey. (All the terms will be explained in the next section).

2 DEFENITIONS AND NOTATION

Definition 2.1. *For a set S and cardinal κ we define:*

1. $2^S := \{X : X \subseteq S\}$
2. $[S]^\kappa := \{X \subseteq S : |X| = \kappa\}$
3. $[S]^{<\kappa} := \{X \subseteq S : |X| < \kappa\}$

In the text ω will denote the set of naturals. In particular 2^ω denotes the power set of the naturals. We see it's a topological space with the usual product (Tychonoff) topology. Any subset of naturals defines an infinite sequence by it's indicator function. So, the power set of naturals can be seen as $\prod_{i<\omega} \mathbf{Z_2}$ ($\mathbf{Z_2}$

denotes the set {0, 1}.
Tychonoff topology is generated by the basis:

$$B = \{\prod_{i<\omega} A_i : A_i = \mathbf{Z_2} \text{ except finitely many } i\}$$

Let us note few topological facts (which can easily be checked by hand) which may be later used in the proof.

1. Note, that this space is induced by a a metric $\mathbf{d}$ defined by

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{1}{n+1} & : x \neq y \\ 0 & : otherwise \end{cases} \tag{1}$$

   while $n \in \omega$ is the minimal index s.t. $x(n) \neq y(n)$ (remember that x and y are infinite sequences of 0's and 1's).

2. We can take another basis for Tychonoff topology in form $\{a_1 \dots a_n\} \times \prod_{n<i<\omega} \mathbf{Z_2}$ where $\{a_1 \dots a_n\}$ is some finite sequence of 0's and 1's.

3. For any infinite set of naturals M, we can define topology 2^M in a similar way to 2^ω. In this case 2^M is homehomorphic to 2^ω (Recall that $\mu : A \to B$ is homeomorphism if it's a continuous bijection and $\mu^{-1} : B \to A$ is also continuous) and to subspace topology on 2^M.[1].

Definition 2.2. *An* n-coloring *of* 2^ω *is a function* $C : 2^\omega \to \{1 \dots n\}$.

Definition 2.3. *A set* $S \subseteq 2^\omega$ *is said to be monochromatic, if there* $i \in \{1 \dots n\}$ *such that* $\forall s \in S : C(s) = i$.

Definition 2.4. *A coloring of* 2^ω *is said to be Ramsey, if* $\exists M \subseteq \omega$ *such that* $[M]^\omega$ *is monochromatic.*

Given a 2-coloring C of 2^ω we may denote S :=$\{X \in 2^\omega | C(X) = 0\}$. If the coloring was Ramsey, then there is $\exists M \subseteq \omega$ such that either $[M]^\omega \subseteq S$ or else $[M]^\omega \subseteq 2^\omega - S$. This allows us to define Ramsey Sets.

Definition 2.5. *A set* $S \subseteq 2^\omega$ *is said to be Ramsey if* $\exists M \subseteq \omega$ *such that either* $[M]^\omega \subseteq S$ *or else* $[M]^\omega \subseteq 2^\omega - S$.

For the people who don't know what 'Borel set' means, the next definition (from Wikipedia) should be enough to understand what the proof is all about. If it didn't help, they are invited to check Wiki (or open any book in functional analysis and glance through first or second chapter) for more info.

Definition 2.6 (σ-algebras and Borel Sets)**.** *A* σ*-algebra over a set* X *is a nonempty collection* Σ *of subsets of* X *(including* X *itself) that is closed under*

[1]We can define product of Z_2 with indicies in M or just add fixed term of 0 at places indicies that don't belong to M. In both case we get the topology homeomorphic to Tychonoff

complementation and countable unions of its members. It is an algebra of sets, completed to include countably infinite operations.

Borel set is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement.

For a topological space X, the collection of all Borel sets on X forms a σ-algebra, known as the Borel algebra or Borel σ-algebra. The Borel algebra on X is the smallest σ-algebra containing all open sets.

Unless explicitly said otherwise M's, N's, P's and Q's will always denote infinite subsets of naturals, and X's, Y's and Z's will denote finite subsets of naturals. A^C will usually denote the complement of A.

3 OPEN SETS ARE RAMSEY

In this section we shall prove that open sets are Ramsey. First - a few definitions.

Definition 3.1. *Let $A, B \subseteq \omega$. We say that $A < B$ if $(\forall a \in A\ \forall b \in B : a < b)$ holds.*

Definition 3.2. *An-M extension of X is a set of the form $X \cup N$ where $X < N$ and $N \subseteq M$.*

In the following defintions and lemmas, S is a fixed subset of 2^ω.

Definition 3.3. *M* ***accpets*** *X if for every P that's M-extension of X, $P \in S$. M* ***rejects*** *X if there is no $N \subseteq M$ such that N accepts X.*

Lemma 3.1.

M accepts (rejects) X iff $\{m \in M : \{m\} > X\}$ accepts (rejects) X.

If M accetps (rejects) X, so does every $N \subseteq M$.

For any X and M, there is an $N \subseteq M$ such that N either accepts or rejects X.

Proof. First statement is clear from the defintion of acception. Second is immidiate, since for all $N \subseteq M$, any N extension is also M-extension of X.
For the third statement: suppose there's $N \subseteq M$ that accepts X. Then it will be the required subset that accepts X. Otherwise, there's no subset of M that accepts M, which by defenition means that M rejects X, thus we can take M to be the required subset. □

Lemma 3.2. *There is $M \subseteq \omega$ such that for any $X \subseteq M$, is either accepted or rejected by M.*

Proof. Let M be any infinite set. By Lemma 3.1, there is $M_0 \subseteq M$ that either accepts or rejects $\emptyset$. Choose any $a_0 \in M_0$. Now supposed that $M_i, a_i (i \leq n)$ have allready been defined. Let $A_n := \{a_i : i \leq n\}$. Then we can choose $M_{n+1} \subseteq M_n$ such that $A_n < M_{n+1}$ and M_{n+1} accepts or rejects every finite

$X \subseteq A_n$ by iterating the previous lemma.
Let $X_1, X_2 \ldots X_{2^n}$ be an enumeration of all the subsets of A_n. Now we perform the following:

1. Choose any $A_n < M_{n+1} \subseteq M_n$ that accepts or rejects X_1
2. for $j \leftarrow 2$ to 2^n do:
3. Choose any $M'_{n+1} \subseteq M_{n+1}$ that accepts or rejects X_j.
4. $M_{n+1} \leftarrow M'_{n+1}$

At the end we get a $M_{n+1} \subseteq M_n$ that accepts or rejects every subset of A_n. Now we choose any $a_{n+1} \in M_{n+1}$. Then $A := \{a_n : n \in \omega\}$ is the required set. This is true because for any finite subset of $X \subseteq A$ there is minimal n such that $X \subseteq A_n$. Notice that $A - A_n = \{a \in A : \{a\} > X\}$. By the first lemma A accepts(rejects) X iff $A - A_n$ accepts(rejects) X. But $A - A_n \subseteq M_{n+1}$. By the construction M_{n+1} must either accept or reject X. Because $A - A_n \subseteq M_{n+1}$ and infinite by first lemma, it either accepts or rejects X. Thus A either accepts or rejects X for any finite $X \subseteq A$. □

Lemma 3.3. *Let M be as in the previous Lemma, and let $X \subseteq M$. If M rejects X, then M rejects any $X \cup \{n\}$ for all but finitely many $n \in M$.*

Proof. Suppose there are infinitely many $n \in M$ such that M doesn't reject, and therefore accepts $X \cup \{n\}$. Let $N := \{n \in M :$ M accepts $X \cup \{n\}\}$. Every N-extension is also M extension of $X \cup \{n\}$ for some $n \in N$. It follows that N accepts X, so M doesn't reject X, a contradiction. □

Lemma 3.4. *Let M be as in the previous Lemma. If M rejects $\emptyset$, then there is $N \subseteq M$ such that N rejects every $X \subseteq N$.*

Proof. Suppose that we've choosen a_i, $i < n$ such that M rejects any $X \subseteq A_n = \{a_i : i < n\}$. Then, by Lemma 3.3, we can choose $a_n \in M$ so that $a_i < a_n$, for $i < n$ and M rejects $X \cup \{a_n\}$ for every $X \subseteq A_n$. (This is true since for any $X \subseteq A_n$ there're finitely many n's such that M doesn't reject $X \cup \{n\}$, and there're finitely many X's). Take $N := \{a_n : n \in \omega\}$. Then M and therefore N rejects every $X \subseteq N$. □

Now we are ready to prove that open sets are Ramsey.

Theorem 3.1. *Open sets are Ramsey.*

Proof. Let $S \subseteq 2^\omega$ be open. By Lemma 3.2, there is M such that M accepts or rejects every $X \subseteq M$. If M accepts $\emptyset$ then $[M]^\omega \subseteq S$.
Otherwise M rejects $\emptyset$. Then by Lemma 3.4 there's $N \subseteq M$ such that N rejects every $X \subseteq N$. We will show that $[N]^\omega \subseteq 2^\omega - S$. Assume the contrary: Then there is $P \subseteq N$ s.t $P \in S$.
S is open. Recall that it means that if $S \neq \emptyset$ (otherwise it would be trivial - all the elements be in $2^\omega - S$), then S is a union of basis elements. As it's been

stated in the previous section, all the basis elements are in form $\prod_{i<\omega} A_i, A_i = \mathbf{Z_2}$ *except finitely many* i. Let $B \subseteq S$ be a base element such that $P \in B$. Then there is $p \in P$ such that for all $k > p$, $A_k = Z_2$. Thus for every Q, if Q and P have the same intersection with $\{n : n \leq p\}$, then $Q \in B$ thus $Q \in S$. But this means that N accepts $\{n \in P : n \leq p\}$ contradicting the fact that N rejects all of its finite subsets. □

4 BOREL SETS ARE RAMSEY

Definition 4.1. *A set $S \subseteq 2^\omega$ is **completely Ramsey** if $f^{-1}(S)$ is Ramsey for every continuous mapping $f : 2^\omega \to 2^\omega$.*

We proceed to prove that Borel sets are Ramsey. To do so, it will be sufficient to show that completely Ramsey sets are $\sigma - algebra$ containing all the open sets, from which it will follow that Borel sets are completely Ramsey, thus Ramsey. First we prove that open sets are completely Ramsey and completely Ramsey sets are closed under complement.

Lemma 4.1. *Every open set is completely Ramsey, since for every open set U and every continous $f : 2^\omega \to 2^\omega$ $f^{-1}(U)$, is open. (This is the definition of continuous function in topology).*

Lemma 4.2. *The complement of completely Ramsey set is completely Ramsey.*

Proof. Let S be completely Ramsey. Then $f^{-1}(S^C) = (f^{-1}(S))^C$. But $f^{-1}(S)$ is Ramsey, which means that it's complement which equals to $f^{-1}(S^C)$ is also Ramsey, thus S^c is completely Ramsey. □

If we show now that completely Ramsey sets are closed under infinte union, then they make $\sigma - algebra$ containing all the open sets, thus the Borel algebra, and we're done.

In the next few lemmas, big capital letters denote subsets of naturals, unless stated otherwise explicitely.

Remember from the second section, that if M is infinite subset of ω, then the one-to-one correspondence between ω and M induces a homeomorphism between them (2^M has also Tychonoff topology, defined in a similar way to 2^ω). Given a finite $X \subseteq \omega$ we can define continuous $g : 2^M \to 2^\omega$ by $g(A) = X \cup A$ for every $A \in 2^M$. To show that g is continuous it's enough to show that for every $P \in 2^M$ and basis element $B \subseteq 2^\omega$ such that $g(P) \in B$ there's a basis element $A \subseteq 2^M$ s.t $P \in A$ and $g\{A\} \subseteq B$. Let B be a basis element containing $g(P) = X \cup P$. Since B is a basis element, then it's of form $\{b_0 \dots b_n\} \times \prod_{n<i<\omega} \mathbf{Z_2}$. Then for any $Y \in 2^M$ such that $\{p_1 \dots p_n\} = \{y_1 \dots y_n\}$[2], $(Y \cup X) \cap \{i : i \leq n\} = (P \cup X) \cap \{i : i \leq n\} = \{b_i : i \leq n\}$ thus in B. Thus for $A = (\{p_1 \dots p_n\} \times \prod_{n<i<\omega} \mathbf{Z_2}) \cap 2^M$, $g(A) \subseteq B$[3].

[2] $\{p_1 \dots p_n\}, \{y_1 \dots y_n\}$ are first n values of their characteristic functions and the equality means their intersections with $\{1 \dots n\}$ are equal.

[3] Recall that the subset topology and product topology on 2^M are homeomorphic, thus all the intersections of 2^M with basis element of 2^ω gives basis for topology on 2^M

Lemma 4.3. *If $S \subseteq 2^\omega$ is completely Ramsey, then, for any finite X and any infinite M, there is $N \subseteq M$ such that either $X \cup P \in S$ for every $P \in [N]^\omega$, or else $X \cup P \notin S$ for every $P \in [N]^\omega$.*

Proof. Let $f : 2^\omega \to 2^M$ be the homeomorphism induced by the one-to-one correspondence between ω and M, and $g : 2^M \to 2^\omega$ as that was defined before. Since $gf : 2^\omega \to 2^\omega$ is continuous and S is completely Ramsey, there is infinite N^* such that either $[N^*]^\omega \subseteq (gf)^{-1}(S)$ or else $[N^*]^\omega \subseteq (gf)^{-1}(S^C)$. Let $N = f(N^*) \in [M]^\omega$; then either $[N]^\omega \subseteq g^{-1}(S)$ or else $[N]^\omega \subseteq g^{-1}(S^C)$. It implies, that either $X \cup P \in S$ for every $P \in [N]^\omega$, or else $X \cup P \in S^C$ for every $P \in [N]^\omega$. □

Lemma 4.4. *If S is completely Ramsey, $M \subseteq \omega$ infinite, X finite subset of M, then there is an N, infinite subset of M, such that $X \subseteq N$ and $S \cap [N]^\omega$ is open (in the relative topology) of $[N]^\omega$.*

Proof. We can obtain an $N_0 \in [M - X]^\omega$ such that, for each $Y \subseteq X$, either $Y \cup P \in S$ for every $P \in [N_0]^\omega$ or else $Y \cup P \notin S$ for every $P \in [N_0]^\omega$, by applying lemma 4.3 repeatedly. (Since $|X|$ is finite, there's finite number n of subsets of X. We denote them by Y_i. Now we perform the following:

1. Using lemma 4.3 choose $N_0 \in [M - X]^\omega$ such that either $Y_1 \cup P \in S$ for every $P \in [N_0]^\omega$ or else $Y_1 \cup P \notin S$ for every $P \in [N_0]^\omega$("required property" until the end of the proof)
2. for $i \leftarrow 2$ to m do:
3. Choose any $N_0' \subseteq N_0$ with the required property with Y_i. If N_0 has the required property for any Y_j for $j < i$ so does it's every infinite subset, thus N_0'. (Because any infinite $P \subseteq N \subseteq N_0$ is also infinite subset of N_0).
4. $N_0 \leftarrow N_0'$

Note that at each iteration i, N_0 had the required property for every Y_j with $j \leq i$ before and after the iteration. Thus it has the required property at the end of the loop.) Let $N = X \cup N_0$. If $P \in S \cap [N]^\omega$, $Q \in [N]^\omega$, and $P \cap X = Q \cap X$, then $Q \in S$. (This is true, because by the construction if $Y \cup P \in S$ for some $Y \subseteq X$, $P \in [N]^\omega$ then it's for every $P \in [N]^\omega$. But for $P - X$, $(P \cap X) \cup (P - X) = P \in S$) from this follows that for every $P \in [N]^\omega \cap S$, there is some basis element of form $(X \cap P) \times 2^{N_0}$ containing it, thus

$$S \cap [N]^\omega \subseteq \Bigg(\bigcup_{\substack{Y \subseteq X \\ Y \times [N_0]^\omega \subseteq S \cap [N]^\omega}} Y \times 2^{N_0} \Bigg).$$

On the other hand it's clear that

$$\Bigg(\bigcup_{\substack{Y \subseteq X \\ Y \times [N_0]^\omega \subseteq S \cap [N]^\omega}} Y \times 2^{N_0} \Bigg) \subseteq S \cap [N]^\omega$$

It follows that $S \cap [N]^\omega$ is open in $[N]^\omega$ $\square$

Lemma 4.5. *If S_n is completely Ramsey for every $n \in \omega$, then, for any infinte $M \subseteq \omega$ there is an $N \in [M]^\omega$, such that $S_n \cap [N]^\omega$ is open in $[N]^\omega$ for every $n \in \omega$.*

Proof. The idea is to iterate the previous lemma. By lemma 4.4 we can choose $N_0 \in [M]^\omega$ so that $S_0 \cap [N_0]^\omega$ is open in $[N_0]^\omega$, and we choose $a_0 \in N_0$. If N_i and a_i have been chosen for $i \leq n$, so that $\{a_i : i \leq n\} \subseteq N_n \in [\omega]^\omega$ and $N_i \cap S_i$ is open in N_i, we choose N_{i+1} so that $\{a_i : i \leq n\} \subseteq N_{n+1} \in [N_n]^\omega$ and $S_{n+1} \cap [N_{n+1}]^\omega$ is open in $[N_{n+1}]^\omega$ (which is possible by Lemma 4.4), and we choose $a_{n+1} \in N_{n+1} - \{a_i : i \leq n\}$. Then $N = \bigcap\{N_n : n \in \omega\}$ works because $\{a_i : i < \omega\} \subseteq N$ by the construction (thus we had to choose a_i at every iteration to assure that we get an infinte set at the end) and since $\forall n \in \omega : [N]^\omega \subseteq [N_n]^\omega$ and $S_n \cap [N_n]^\omega$ is open in $[N_n]^\omega$ thus $S_n \cap [N]^\omega$ is open for all $n \in \omega$, $\square$

Lemma 4.6. *If S_n is completely Ramsey for every $n \in \omega$, then $\bigcup S_n : n \in \omega$ is completely Ramsey.*

Proof. Let $S = \bigcup\{S_n : n \in \omega\}$. By Lemma 4.5 there is an $N \in [\omega]^\omega$ such that $S_n \cap [N]^\omega$ is open in $[N]^\omega$ for every $n \in \omega$. Hence $S \cap [N]^\omega$ is open in $[N]^\omega$; i.e., $S \cap [N]^\omega = T \cap [N]^\omega$ for some open $T \subseteq 2^\omega$. Note that T is completely Ramsey by Lemma 4.1, So by Lemma 4.3, there is $P \in [N]^\omega$ such that either $[P]^\omega \subseteq T$ or else $[P]^\omega \subseteq 2^\omega - T$ (Substitute sets in Lemma 4.3 with; $X = \emptyset$, $N = P$, $M = N$); thus either $[P]^\omega \subseteq S$ or else $[P]^\omega \subseteq 2^\omega - S$. This shows that S is Ramsey. Now if $f : 2^\omega \to 2^\omega$ is continous then every $f^{-1}(S_n)$ is completely Ramsey, and $f^{-1}(S) = \bigcup\{f^{-1}(S_n) : n \in \omega\}$. By foregoing argument, $f^{-1}(S)$ is Ramsey which proves that S is completely Ramsey. $\square$

Theorem 4.1. *Every Borel set is Ramsey.*

Proof. By the lemmas 4.1, 4.2, 4.6 the class of completely Ramsey functions is $\sigma - algebra$ which includes all the open sets, therefore all the Borel sets. $\square$

5 A GENERALIZATION TO FINTITE COLORINGS

We now generalize to any finite coloring. (the idea to do it is due to Boaz Tzaban). Note that the defintion of Ramsey sets in the 2nd section remains valid for subsets of $[\omega]^\omega$.

Definition 5.1. *A function $f : X \to Y$ is borel if for any Borel set $B \subseteq Y$, $f^{-1}(B)$ is Borel in X.*

Next statement is based on the facts that given a subspace $Y \subseteq X$, $B \subseteq Y$ is open iff $B = B' \cap Y$ for open set $B' \subseteq X$, and that Borel algebra is the minimal σ-algebra that contains all open sets.

Lemma 5.1. *Let X be a topological space, $Y \subseteq X$ subspace. $B \subseteq Y$ is Borel set iff $B = B' \cap Y$ for some Borel set $B' \subseteq X$.*

Now we prove another lemma that will help us to generalize to finite colorings.

Lemma 5.2. *Let $B \subseteq [\omega]^\omega$ be Borel in $[\omega]^\omega$. Then B is Ramsey.*

Proof. Since B is Borel set, it follows from the previous statement there is $B' \subseteq 2^\omega$ Borel set, such that $B = B' \cap [\omega]^\omega$. Since B' is Borel it is Ramsey, thus there is $M \in [\omega]^\omega$ such that either $[M]^\omega \subseteq B'$ or $[M]^\omega \subseteq 2^\omega - B'$. But $[M]^\omega$ is a subset of $[\omega]^\omega$, thus $[M]^\omega \subseteq B = B' \cap [\omega]^\omega$ holds or $[M]^\omega \subseteq [\omega]^\omega - B = (2^\omega - B') \cap [\omega]^\omega$ holds. Thus B is Ramsey. □

Theorem 5.1. *Any Borel coloring of $[\omega]^\omega$ is Ramsey.*

Proof. The proof is by induction. For 2 - coloring it's true by the previous lemma.
We assume that the theorem is true for all $k < n$ and prove for n. Let $C : [\omega]^\omega \to \{1 \dots n\}$ be a Borel coloring[4] of $[\omega]^\omega$. Then $C^{-1}\{1\}$ is Borel, thus Ramsey, and thus there is $[M]^\omega$ such that $[M]^\omega \subseteq C^{-1}\{1\}$ or $[M]^\omega \subseteq [\omega]^\omega - C^{-1}\{1\}$. If $[M]^\omega \subseteq C^{-1}\{1\}$ holds we're done. Otherwise $C|_{[M]^\omega} = C'$ is Borel coloring of $[M]^\omega$ with n - 1 colors. Note that $[\omega]^\omega$ and $[M]^\omega$ are homeomorphic. Thus the induction hypothesis holds for $[M]^\omega$ and there is monochromatic $[N]^\omega \subseteq [M]^\omega$. But it's also subset of $[\omega]^\omega$. From this follows that C is Ramsey. □

[4]This means C is Borel function from $[\omega]^\omega$ to {1 ... n}