ON QUADRUPLES OF CONSECUTIVE kth POWER RESIDUES

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In a recent paper of D. H. and Emma Lehmer [2], the function $\Lambda(k, m)$ was defined (for arbitrary integers k and m) as follows:

Let p be a sufficiently large prime and let $r = r(k, m, p)$ be the least positive integer such that

$$r, r+1, r+2, \cdots, r+m-1$$

are all congruent modulo p to kth powers of positive integers. Define

$$\Lambda(k, m) = \limsup_{p \to \infty} r(k, m, p).$$

In [2] it was shown that $\Lambda(k, 4) = \infty$ for $k \leqq 1048909$ and it was conjectured that $\Lambda(k, 4) = \infty$ for all k. In this paper we establish this conjecture with the following

THEOREM. $\Lambda(k, 4) = \infty$.

PROOF. It suffices to prove the theorem for values of k which are prime. The proof makes use of the following proposition which is a special case of a result of Kummer [1] (see also [3]).

PROPOSITION. *Let k be a prime and let $\gamma_1, \cdots, \gamma_n$ be an arbitrary sequence of kth roots of unity. Then there exist infinitely many primes p with corresponding kth power character χ modulo p such that*

$$\chi(p_i) = \gamma_i, \qquad 1 \leqq i \leqq n,$$

where p_i denotes the ith prime.

Thus, for any n and prime k, there exists a prime p with corresponding kth power character χ modulo p such that

$$\chi(2) \neq 1,$$
$$\chi(p_i) = 1, \qquad 2 \leqq i \leqq n.$$

Now consider any four consecutive positive integers all less than p_n. It is clear that exactly one of these integers must equal $2(2d+1)$ for some integer d. But we have

$$\chi(2(2d+1)) = \chi(2)\chi(2d+1) = \chi(2)\cdot 1 \neq 1$$

since $2d+1$ is the product of odd primes less than p_n. Therefore

Received by the editors December 29, 1962.

$2(2d+1)$ is not a kth power residue modulo p. Since n was arbitrary then $\Lambda(k, 4)=\infty$. This proves the theorem.

References

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ON DECOMPOSITIONS OF PARTIALLY ORDERED SETS

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1. **Introduction.** Let P be a set which is partially ordered by a relation $\leqq$. A *decomposition* $\mathfrak{D}$ of P is a family of mutually disjoint non-empty chains in P such that $P=\bigcup\{C: C\in\mathfrak{D}\}$. Two elements x, y of P are *incomparable* if and only if $x \not\leqq y$ and $y \not\leqq x$. A *totally unordered* set in P is a subset in which every two different elements are incomparable. We denote the cardinal number of a set S by $|S|$.

Dilworth [1] has proved the following well-known decomposition theorem.

Theorem 1 (Dilworth). *Let P be a partially ordered set, and suppose that n is a positive integer such that*

$$n = \max\{|A| : A \text{ is a totally unordered subset of } P\}.$$

Then there is a decomposition $\mathfrak{D}$ of P with $|\mathfrak{D}|=n$.

It is natural to ask whether, in this theorem, the positive integer n may be replaced by an infinite cardinal number. However, the theorem is no longer valid in this case, as is shown by an example in [3] which is due in essence of Sierpinski [2]. In this example P is a set of pairs which represents a 1-1 mapping from ω_1, the first uncountable ordinal, into the real numbers. $(x_1, y_1)\leqq(x_2, y_2)$ is defined by: $x_1\leqq x_2$ (as ordinals) and $y_1\leqq y_2$ (as real numbers). The purpose of this note is to show that a similar idea leads, given any infinite cardinal k, to

Received by the editors December 28, 1962.