Algorithms for Maximal Ind. Set

Exposition by William Gasarch
This talk is based on parts of the AWESOME book

Exact Exponential Algorithms
by
Fedor Formin and Dieter Kratsch
**Definition:** If $G = (V, E)$ is a graph then $I \subseteq V$ is an Ind. Set if $(\forall x, y \in V)[(x, y) \notin E]$. The set $I$ is a MAXIMUM IND SET if it is an Ind Set and there is NO ind set that is bigger.

**Goal:** Given a graph $G$ we want the SIZE of the Maximum Ind. Set. Obtaining the set itself will be an easy modification of the algorithms which we will omit.

**Abbreviation:** MIS is the Maximum Ind Set problem.

**BILL**- Do examples and counterexamples on the board.
Why Do We Care About MIS?

1. MIS is NP-complete.

2. MIS comes up in applications (so my friends in systems tell me).
1. Will we show that MIS is in P?

NO. Too bad. If we had $1,000,000 then we wouldn’t have to worry about whether the REU grant gets renewed.

2. We will show algorithms for MIS that
2.1 Run in time $O(\alpha^n)$ for various $\alpha < 1$. NOTE: By $O(\alpha^n)$ we really mean $O(p(n)^\alpha n)$ where $p$ is a poly. We ignore such factors.
2.2 Quite likely run even better in practice.
1. Will we show that MIS is in P?
   NO.
1. Will we show that MIS is in P?
   NO.
   Too bad.
OUR GOAL

1. Will we show that MIS is in P?

   NO.

   Too bad.

   If we had $1,000,000 then we wouldn’t have to worry about whether the REU grant gets renewed.
1. Will we show that MIS is in P?
   
   NO.
   
   Too bad.
   
   If we had $1,000,000 then we wouldn’t have to worry about whether the REU grant gets renewed.

2. We will show algorithms for MIS that
   
   2.1 Run in time $O(\alpha^n)$ for various $\alpha < 1$. NOTE: By $O(\alpha^n)$ we really mean $O(p(n)\alpha^n)$ where $p$ is a poly. We ignore such factors.
   
   2.2 Quite likely run even better in practice.
If all of the degrees are $\leq 2$ then the problem is EASY.
BILL- HAVE THEM DO THIS.
If $G = (V, E)$ is a graph and $v \in V$ then

$$N[v] = \{v\} \cup \{u \mid (v, u) \in E\}.$$}

The NEIGHBORS of $v$ AND $v$ itself.
MIN DEG ALGORITHM

ALG\( (G = (V, E) \): \text{A Graph})

\[ v = \text{ vertex of min degree } \]
\[ \text{for } u \in N[v] \]
\[ m_u = ALG(G - N[m_u]) \]
\[ m = \min \{ m_u \mid u \in N[v] \} . \]
\[ \text{RETURN} (1 + m) \]

**BILL:** TELL CLASS TO FIGURE OUT WHY WORKS.
Let $N[v] = \{v, x_1, \ldots, x_{d(v)}\}$.

\[
T(n) \leq 1 + T(n - d(v) - 1) + \sum_{i=1}^{d(v)} T(n - d(x_i) - 1)
\leq 1 + T(n - d(v) - 1) + \sum_{i=1}^{d(v)} T(n - d(v) - 1)
\leq 1 + (d(v) + 1) T(n - (d(v) + 1))
\]

BILL: HAVE CLASS ANALYSE $T(n) = 1 + sT(N - s)$. THEN DO ON BOARD.
HOW GOOD?

1. Runs in $T(n) = O((3^{1/3})^n) \leq O((1.42)^n)$.
2. Works well on high degree graphs until they become low degree graphs.
4. Makes more sense to take High degree nodes.
MAX DEG ALG

ALG(G)

1. If (∃v)[d(v) = 0] then RETURN(1 + ALG(G − v)).
2. If (∃v)[d(v) = 1] then RETURN(1 + ALG(G − N[v])).
3. If (∀v)[d(v) ≤ 2] then CALL 2-MIS ALG.
4. If (∃v)[d(v) ≥ 3] then
   4.1 Let v* be of max degree
   4.2 Return MAX of 1 + ALG(G − N[v*]), ALG(G − v*).

BILL- HAVE CLASS DISCUSS WHY WORKS.
\[ T(n) \leq T(n - d(v) - 1) + T(n - 1) \]
\[ T(n) \leq T(n - 4) + T(n - 1) \]

Guess \( T(n) = \alpha^n \)

\[ \alpha^n = \alpha^{n-4} + \alpha^{n-1} \]

\[ \alpha^4 = 1 + \alpha \]

\[ \alpha^4 - \alpha - 1 = 0 \]

\( \alpha \sim 1.38. \)
HOW GOOD?

1. Runs in $T(n) = O((1.38)^n)$.
2. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.
3. WORKS really well in practice.
1. Runs in $T(n) = O((1.38)^n)$.
2. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.
3. WORKS really well in practice.

It works in practice—can we make it work in theory?
Need to MEASURE progress better.

1. We measure a node of degree $\leq 1$ as having weight ZERO.
2. We measure a node of degree 2 as having weight $\frac{1}{2}$.
3. We measure a node of degree $\geq 3$ as having weight ONE.

SO we view $|V|$ as

$$\frac{1}{2}(\text{number of verts of degree 2}) + (\text{number of verts of degree 3})$$

We still refer to this as $n$. 
Have picked $v^*$. 

1. Assume there are no vertices of degree $\leq 1$ (else would not be in $v^*$ case) 
2. Assume $v^*$ has $d_2$ vertices of degree 2. 
3. Assume $v^*$ has $d_3$ vertices of degree 3. 
4. Assume $v^*$ has $d_{\geq 4}$ vertices of degree $\geq 4$. 

Exposition by William Gasarch

Algorithms for Maximal Ind. Set
**Better Analysis of G − N[v] Case**

\(G − N[v^*]:\)

1. Loss of \(v^*\) is loss of 1.
2. Loss of \(d_2\) vertices of degree 2: Loss is \(\frac{d_2}{2}\).
3. Loss of \(d_3\) vertices of degree 3: Loss is \(d_3\).
4. Loss of \(d_{\geq 4}\) vertices of degree \(\geq 4\): Loss is \(d_{\geq 4}\).

Total Loss: \(1 + \frac{d_2}{2} + d_3 + d_{\geq 4}\).

Work to do:

\[T(n - (1 + \frac{d_2}{2} + d_3 + d_{\geq 4}))\]
BETTER ANALYSIS OF $G - v$ CASE

$G - v^*$:

1. Loss of $v^*$ is loss of 1.
2. The $d_2$ verts of deg 2 become $d_2$ verts of deg $\leq 1$. Loss is $\frac{d_2}{2}$.
3. The $d_3$ verts of deg 3 become $d_3$ verts of deg $\leq 2$. Loss is $\frac{d_3}{2}$.
4. The $d_{\geq 4}$ verts of deg $\geq 4$. No Loss.

Total Loss: $1 + \frac{d_2}{2} + \frac{d_3}{2}$.

Work to do:

$$T(n - (1 + \frac{d_2}{2} + \frac{d_3}{2}))$$
\[
T(n) \leq T(n - (1 + \frac{d_2}{2} + d_3 + d_{\geq 4})) + T(n - (1 + \frac{d_2}{2} + \frac{d_3}{2})) \\
\leq T(n - 1) + T(n - (1 + d_2 + \frac{3d_3}{2} + d_{\geq 4})) \\
\leq T(n - 1) + T(n - (d(v^*) + 1))
\]

1. If \(d(v^*) \geq 4\) then get

\[
T(n) \leq T(n - 1) + T(n - 5)
\]

BILL- HAVE STUDENTS DO.

2. If \(d(v^*) = 3\) then BILL- HAVE STUDENTS DO.
HOW GOOD?

1. Runs in $T(n) \leq O((1.3248)^n)$.

2. Using cleverer choice of weights can get $O((1.2005)^n)$. (Deg2 nodes weigh 0.596601, Deg3 nodes weigh 0.928643, Deg4 nodes weigh 1.)

3. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.

4. WORKS really well in practice, and this analysis may say why.
Best known runs in time

$$O((1.2109)^n).$$

1. Order constant is REASONABLE.
2. LOTS of cases depending on degree.
3. Sophisticated analysis.
4. Good in practice? A project for NEXT YEARS REU!!!!